THE SOLUTION OF FINAL OF ALGEBRA

1. (1) \( \phi(f + g) = \int_0^1 (f + g)(x)dx = \int_0^1 [f(x) + g(x)]dx \)
   \( = \int_0^1 f(x)dx + \int_0^1 g(x)dx = \phi(f) + \phi(g) \).

(2) the index of the subgroup \( \langle \mu \rangle \) in \( S_6 \) is \( \frac{6!}{4} = 180 \).

2. (1) \( \phi(10, 3) = \phi(10(1, 0) + 3(0, 1)) = \phi(1, 0)^{10} \cdot \phi(0, 1)^3 \)
   \( = [(1, 7)(6, 10, 8, 9)]^2(3, 5)(2, 4) = (6, 8)(10, 9)(3, 5)(2, 4) \).

(2)

3. (1) Let \( \phi : Z_5 \rightarrow Z \) be a nontrivial (group) homomorphism with \( \phi(1) = a, a \neq 0 \),
   we have \( 5\phi(1) = \phi(5 \cdot 1) = \phi(0) = 0 \)
   \( \Rightarrow \phi(1) = a = 0 \)
   hence there is no nontrivial homomorphism.

(2) Let the order of \( ab \) is \( n \)
   \( ab \cdot ab \cdots ab = e \)
   \( \Rightarrow b \cdot (ab \cdot ab \cdots ab) \cdot a = bea = ba \)
   \( \Rightarrow (ba)^{n+1} = ba \)
   \( \Rightarrow (ba)^n = e \)
   the order of \( ba \leq n \). If the order of \( ba < n \), then the order of \( ab < n \).
   That is contradiction. Hence the order of \( ab \) is equal to the order of \( ba \).

4. (1) \( * : G \times X \rightarrow X \)
   \( *(e, H) = eHe^{-1} = H \)
   and \( \forall g_1, g_2 \in G \)
   \( *(g_1g_2H) = (g_1g_2)H(g_1g_2)^{-1} = g_1g_2Hg_2^{-1}g_1^{-1} \)
   \( = g_1 * (g_2H)g_1^{-1} = *(g_1 * (g_2H)) \)
   hence \( X \) is a \( G \)-set

(2) \( \Rightarrow \) Let \( O(H) = \{ H \} \)
   \( \forall g \in G, \forall h \in H = O(H) \)
   \( \Rightarrow ghg^{-1} \in H \)
   \( \Rightarrow H \) is normal in \( G \)
   \( \Leftarrow \) It’s clear that \( H \subseteq O(H) \)
   \( \forall g \in G, \forall h \in O(H) \Rightarrow ghg^{-1} \in H \)
   \( \Rightarrow h \in gHg^{-1} = H \)
   \( \Rightarrow O(H) = H \)

5. (1) The characteristic of \( M_2(Z_n) \) is \( n \)

(2) \( 1 - 2a \in R \)
   \( (1 - 2a)^2 = 1 - 2a \)
   \( 1 - 4a + 4a^2 = 1 - 2a \)
   \( \Rightarrow a = 0 \)
   \( 1 - 2a = 1, \) that is an unit of \( R \)
6. (1) Let $H$ be the set of all idempotent elements of a commutative ring $R$
\[ \forall a, b \in H, \ a^2 = a \text{ and } b^2 = b \]
\[ (ab)^2 = a^2b^2 = ab \]
\[ \Rightarrow ab \in H \]
$H$ is closed under multiplication.

(2) \{ (0, 1), (0, 3), (0, 4), (0, 0), (1, 1), (1, 3), (1, 4), (1, 0) \}

7. (1) Let $\varphi : D \Rightarrow \mathbb{Q}(D)$ with $\varphi(a) = (a, 1)$
then we can show that $\varphi$ is an isomorphism

(2) There is no zero in $\mathbb{Z}_7$

8. (1) $N = \{ f(x) \in R[x] \mid f(2) = 0 \}$
\[ \forall f(x), g(x) \in H, \ f(2) = g(2) = 0 \Rightarrow f(2) - g(2) = 0 \]
\[ \Rightarrow f(x) - g(x) \in N \]
and $\forall h(x) \in R[x], \ (h \cdot f)(2) = h(2)f(2) = 0, \ (f \cdot h)(2) = f(2)h(2) = 0$
\[ \Rightarrow h(x)f(x) \in H, \ f(x)h(x) \in H, \text{ hence } H \text{ is an ideal of } R[x] \]

(2) Let $N = \langle x - 2 \rangle$
\[ R[x]/\langle x - 2 \rangle = R \text{ is a field.} \]
hence $N$ is a maximal ideal of $R[x]$

9. (1) Let $f(x) = x^p + a \in \mathbb{Z}_p[x]$
If $a = 0 \Rightarrow f(x) = x^p \Rightarrow f(0) = 0$, then $f(x)$ is reducible
If $a \neq 0 \Rightarrow f(-a) = (-a)^p + a = 0$, then $f(x)$ is reducible
\[ (\forall a \in \mathbb{Z}_p, (-a)^{p-1} = e \Rightarrow (-a)^p = (-a)) \]

(2) $\forall a, b \in R$
\[ \phi_p(a + b) = (a + b)^p = a^p + b^p = \phi_p(a) + \phi_p(b) \]
(the characteristic of $R$ is $p$)
\[ \phi_p(ab) = (ab)^p = 0 \text{ and } \phi_p(a)\phi_p(b) = 0 \cdot 0 = 0 \]
hence $\phi_p(a) = a^p$ is a (ring) homomorphism.

10. Let $\phi_m$ be the endomorphism of the abelian group such that $\phi_m(1) = m$.
Then $\{ \phi_m \mid m \in \mathbb{Z} \}$ is the entire homomorphism ring.
Define $\varphi : \text{End}(\mathbb{Z}) \rightarrow \mathbb{Z}$ by $\varphi(\phi_m) = m$
Then we can show that $\varphi$ is an isomorphism