1. 

(1) Since \( \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0 \) \( \Rightarrow \) \( u \) is harmonic.

(2) \( \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y} = -3y^2 + 3x^2, \ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -6xy \Rightarrow v(x, y) = x^3 - 3xy^2 + C. \)

2. 

(1) \( f \) is analytic at \( z = 0 \) and \( f'(0) = 2 \neq 0 \) \( \Rightarrow \) \( f \) is conformal at \( z = 0. \)

(2) \[ \cos(z + 2\pi) = \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} = \frac{e^{2\pi i}e^{iz} + e^{-2\pi i}e^{-iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos z. \]

3. 

(1) \( (\sqrt{3} + i)^3(1 - \sqrt{3}i)^2 \)

\[ = [2(\frac{\sqrt{3}}{2} + \frac{i}{2})][2(\frac{1}{2} - \frac{\sqrt{3}}{2}i)]^2 \]

\[ = 2^5[\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}]^3[\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}] \]

\[ = 2^5i(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) \]

\[ = 16\sqrt{3} - 16i. \]

(2) \( (-1)^i = e^{i \log(-1)} = e^{i(2n+1)\pi} = e^{-(2n+1)\pi}. \)

4. 

(1) Since \( e^{\frac{y}{z^2 + 9}} \) is analytic at \( |z| \leq 2 \) \( \Rightarrow \) \( \oint \frac{e^{x}}{(z+3)(z-3)} \, dz = 0. \)

(2) \( \oint \frac{e^{x}}{(z+3)(z-3)} \, dz = \oint \frac{e^{x}}{z-3} \, dz = 2\pi i \frac{e^{3}}{3+3} = \frac{\pi i}{3} e^{3}. \)

5. 

(1) \( \gamma(t) = \begin{cases} e^{i(\frac{2}{3} - t)} & 0 \leq t \leq \frac{1}{2} \\ 2t + (2t - 1)i & \frac{1}{2} \leq t \leq 1 \end{cases} \)

(2) \( f(x + iy) = x^2 + iy^2, \)

\( z(t) = (1 - t) + ti, \)

\( z'(t) = -1 + i, \)

\( f(z(t)) = (1 - t)^2 + it^2 \)

\( \Rightarrow \oint f(z) \, dz = \int [(1 - t)^2 + it^2](-1 + i) \, dt = -\frac{2}{3}. \)

6. 

If \( f(z) = u(x, y) + iv(x, y) \)

\( g(z) = u^2(x, y) - v^2(x, y) - 2iu(x, y)v(x, y) \)

\[ \frac{\partial (\text{Re} \ g)}{\partial x} = \frac{\partial}{\partial x} u^2(x, y) - \frac{\partial}{\partial x} v^2(x, y) \]

\[ = 2u(x, y) \frac{\partial}{\partial x} u(x, y) - 2v(x, y) \frac{\partial}{\partial x} v(x, y) \]

\[ = 2u(x, y) \frac{\partial}{\partial x} u(x, y) + 2v(x, y) \frac{\partial}{\partial x} u(x, y) \]

\[ = \frac{\partial}{\partial y} [-2u(x, y)v(x, y)] \]

\[ = \frac{\partial (\text{Im} \ g)}{\partial y}. \]

Similarly \( \frac{\partial (\text{Re} \ g)}{\partial y} = -\frac{\partial (\text{Im} \ g)}{\partial x} \) \( \Rightarrow \) \( g \) is analytic.

7. 

\[ |e^{iz}| = |e^{ix-y}| = |e^{-y}e^{ix}| = |e^{-y}| = e^{-y} \leq e^2. \]

8. 

Since \( f = u + iv \) is analytic

\[ \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \]
\[ \Rightarrow \frac{\partial f}{\partial z} = \frac{1}{2} \left[ \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right] \]
\[ = \frac{1}{2} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + i \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) \right] \]
\[ = \frac{1}{2} \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] + \frac{1}{2} i \left[ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right] \]
\[ = 0. \]

9. Since \( f(z) \) is an entire function and \( |\frac{f(z)}{z^n}| \leq M \)
by Liouville’s theorem \( \Rightarrow \frac{f(z)}{z^n} \) is a constant, i.e. \( f(z) = cz^n \).

10. \( f \) is analytic in \( D \)
then the function \( \frac{1}{f} \) is analytic in \( D \) and \( |\frac{1}{f}| \geq 0 \)
Apply the maximum modulus theorem to the function \( \frac{1}{f} \),
\( |\frac{1}{f(z)}| \) attains its maximum value on the boundary.
i.e., \( |f(z)| \) attains its minimum value on the boundary.