1. (5 points) Let \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \) be a differentiable vector-valued function such that \( \mathbf{r}'(t) \neq 0 \) for all \( t \geq 0 \). Show that the arc length function \( s \) defined by \( s(t) = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \), for \( t \geq 0 \), has an inverse \( t = \phi(s) \).

**Solution:** Since \( \frac{ds}{dt} = \|r'(t)\| \neq 0 \) for all \( t \geq 0 \), Mean Value Theorem implies that the function \( s \) is an 1–1 function and the inverse \( t = \phi(s) \) exists.

2. (5 points) Consider the circular helix \( \mathbf{r}(t) = 3\cos t + 3\sin tj + 4tk \) for \( t \geq 0 \). Determine the arc length \( s \) as a function of \( t \) by evaluating the integral \( s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \). What is the inverse \( t = \phi(s) \)?

**Solution:** By direct computation, we get \( s = \int_0^t \sqrt{9 + 16} = 5t \), and \( t = \frac{s}{5} \).

3. (5 points) Let \( \mathbf{r}(t) = e^t \cos t + e^t \sin tj + e^t\mathbf{k} \). Find the curvature \( \kappa \) of the path.

**Solution:** By direct computation, we get \( \mathbf{r}'(t) = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t \mathbf{k} \), \( \|\mathbf{r}'(t)\| = \sqrt{3}e^t \), and \( T(t) = \frac{1}{\sqrt{3}}(\cos t - \sin t, \cos t + \sin t, 1) \).

Therefore, \( \kappa = \|\frac{dT}{ds}\| = \|\frac{dT}{dt} \frac{1}{ds/dt}\| = \sqrt{\frac{2}{3}} \frac{1}{\sqrt{3}e^t} = \frac{\sqrt{2}e^{-t}}{3} \).

4. (5 points) Show that the curvature of a polar curve \( r = f(\theta) \) is given by \( \kappa = \frac{|[f(\theta)]^2 + 2[f'(\theta)]^2 - f(\theta)f''(\theta)|}{([f'(\theta)]^2 + [f''(\theta)]^2)^{3/2}} \).

**Hint:** Express the curve as \( \mathbf{R}(\theta) = (x(\theta), y(\theta)) = (r \cos \theta, r \sin \theta) \), where \( r = f(\theta) \).

**Solution:** By direct computation, we get \( \mathbf{R}'(\theta) = (x'(\theta), y'(\theta)), T(\theta) = \frac{1}{\sqrt{(x')^2 + (y')^2}}(x', y') \),
\[
\frac{dT}{d\theta} = -(x'y'' + y'x'')(x', y') + \frac{(x')^2 + (y')^2}{((x')^2 + (y')^2)^{3/2}}(x'', y'') = \frac{x'y'' - y'x''}{((x')^2 + (y')^2)^{3/2}}(x', y'),
\]
\[
\|\frac{dT}{d\theta}\| = \frac{|x'y'' - y'x''|}{(x')^2 + (y')^2}, \text{ and } \kappa = \|\frac{dT}{ds}\| \frac{1}{ds/d\theta} = \frac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{3/2}}.
\]

Now \( x = f \cos \theta, y = f \sin \theta, \) and
\[
x' = f' \cos \theta - f \sin \theta, y' = f' \sin \theta + f \cos \theta, \]
\[
x'' = f'' \cos \theta - 2f' \sin \theta - f \cos \theta, y'' = f'' \sin \theta + 2f' \cos \theta - f \sin \theta,\]

we get \( \kappa = \frac{|[f(\theta)]^2 + 2[f'(\theta)]^2 - f(\theta)f''(\theta)|}{([f'(\theta)]^2 + [f''(\theta)]^2)^{3/2}} \).