A: Linear Algebra and Its Applications by Gilbert Strang, 4th Edition
B: Linear Algebra by Larry Smith, 3rd Edition
C: Introduction to Linear Algebra by Gilbert Strang, 3rd Edition

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numbers, which we denote by $\mathbb{C}$. A vector space with complex scalars is called a **complex vector space**. The axioms for a complex vector space are exactly the same as for a real vector space except that the numbers (= scalars) are to be complex. The generic example of a complex vector space is the complex Cartesian space $\mathbb{C}^k$ of k-tuples $\mathbf{A} = (a_1, \ldots, a_k)$ of *complex* numbers, where for vectors $\mathbf{A} = (a_1, \ldots, a_k)$ and $\mathbf{B} = (b_1, \ldots, b_k)$ and for scalars $r \in \mathbb{C}$, vector addition and scalar multiplication are given by

$$\mathbf{A} + \mathbf{B} = (a_1 + b_1, \ldots, a_k + b_k)$$

and

$$r\mathbf{A} = (ra_1, \ldots, ra_k).$$

In the next few chapters we will study only real vector spaces, indicating where necessary the modifications required in the complex case.

2.4 Exercises

1. Let $A = \{(2a, a) \mid a \in \mathbb{R}\}$, $B = \{(b, b) \mid b \in \mathbb{R}\}$. Find $A \cup B$ and $A \cap B$.

2. Denote by $\mathbb{Z}$ the set of all integers and let $A = \{(2n, n) \mid n \in \mathbb{Z}\}$, $B = \{(k+1, k) \mid k \in \mathbb{Z}\}$. Find $A \cup B$, $A \cap B$.

3. If $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$.

4. If $A \subseteq B$, $A \subseteq C$, then $A \subseteq B \cap C$.

5. If $A \supseteq B$, $A \supseteq C$, then $A \supseteq B \cup C$.

6. Show that $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, where A, B, C are sets.

7. Let $A = \{x \in \mathbb{R} \mid |x| > 1\}$, $B = \{x \in \mathbb{R} \mid -2 < x < 3\}$. Find $A \cup B$ and $A \cap B$.

8. Let $\mathcal{V}$ be a vector space. Prove each of the following statements:

(a) If $\mathbf{A} \in \mathcal{V}$ and a is a number, then $a\mathbf{A} = \mathbf{0}$ if and only if $a = 0$ or $\mathbf{A} = \mathbf{0}$, or both.

(b) If $\mathbf{A} \in \mathcal{V}$ and a is a number, then $a\mathbf{A} = \mathbf{A}$ if and only if $a = 1$ or $\mathbf{A} = \mathbf{0}$, or both.

9. Let $\mathcal{V}$ be the set of all ordered pairs of real numbers (a, b). Define an addition for the elements of $\mathcal{V}$ by the rule

$$(a, b) \oplus (c, d) = (a + c, b + d),$$

and a multiplication of elements of $\mathcal{V}$ by numbers by the rule

$$r \cdot (c, d) = (rc, 0).$$

Is $\mathcal{V}$ with these two operations a vector space? Justify your answer.

10. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Denote by $\mathcal{V} \oplus \mathcal{W}$ the set of all ordered pairs $(\mathbf{A}, \mathbf{B})$, where $\mathbf{A} \in \mathcal{V}$ and $\mathbf{B} \in \mathcal{W}$. Define an addition for the elements of $\mathcal{V} \oplus \mathcal{W}$ by the rule

$$(\mathbf{A}, \mathbf{B}) + (\mathbf{C}, \mathbf{D}) = (\mathbf{A} + \mathbf{C}, \mathbf{B} + \mathbf{D}),$$

and a multiplication of elements of $\mathcal{V}$ by numbers by the rule

$$r \cdot (\mathbf{A}, \mathbf{B}) = (r\mathbf{A}, r\mathbf{B}).$$

Is $\mathcal{V} \oplus \mathcal{W}$ with these two operations a vector space? Justify your answer. (Beware of the fact that + and · are used with multiple meanings in the preceding equations. $\mathcal{V} \oplus \mathcal{W}$ is called the **direct sum** of $\mathcal{V}$ and $\mathcal{W}$.)

11. A **translation** of the plane Π is a function $T : \Pi \longrightarrow \Pi$ with the following two properties:

① There is a constant k, called the **length** of the translation, such that for every point $p \in \Pi$ the distance from p to $T(p)$ is equal to k.

② For any two points $p, q \in \Pi$ the distance from p to q is the same as the distance from $T(p)$ to $T(q)$.

Introduce Cartesian coordinates in Π and show:

(a) If T is a translation, then for every $p = (x, y) \in \Pi$, $T(p) = (x + l_1, x + l_2)$, where $T(0, 0) = (l_1, l_2)$. T is said to be the **translation by** $l = (l_1, l_2)$.

(b) If $l = (l_1, l_2) \in \Pi$, show that

$$T(p) = (x + l_1, y + l_2), \qquad p = (x, y) \in \Pi$$

defines a translation of Π.

If T, S are translations, define their sum $T \oplus S$ by

$$(T \oplus S)(p) = T(S(p)).$$

(c) Show that $T \oplus S$ is again a translation.

If T is the translation by $l = (l_1, l_2)$ and a is a number, define $a \cdot T$ to be the translation by (al_1, al_2).

(d) Show that the set of translations with the addition $\oplus$ and scalar multiplication · is a vector space.

12. Show that if a vector space contains two elements, then it contains infinitely many.

13. Let $\mathbf{A}, \mathbf{B}$ be elements of the vector space $\mathcal{V}$. Show that there exists a unique element $\mathbf{X}$ of $\mathcal{V}$ such that $\mathbf{A} + \mathbf{X} = \mathbf{B}$.

EXAMPLE 3: Let S be the set consisting of the three elements a, b, c and consider all the functions f in $\mathrm{Fun}(S)$ with the property that

$$f(a) + f(b) + f(c) = 0.$$

Call this set of functions $\mathcal{N}$. If $f, g \in \mathcal{N}$ and $r \in \mathbb{R}$, then by the definition of vector addition and scalar multiplication in $\mathrm{Fun}(S)$ we find that

$$\begin{aligned}
&(f+g)(a) + (f+g)(b) + (f+g)(c) \\
&\quad = f(a) + g(a) + f(b) + g(b) + f(c) + g(c) \\
&\quad = f(a) + f(b) + f(c) + g(a) + g(b) + g(c) = 0, \\
&(rf)(a) + (rf)(b) + (rf)(c) = r \cdot f(a) + r \cdot f(b) + r \cdot f(c) \\
&\quad = r\,(f(a) + f(b) + f(c)) = r \cdot 0 = 0,
\end{aligned}$$

and therefore $\mathcal{N}$ is a linear subspace of $\mathrm{Fun}(S)$.

You might want to think through what happens in Examples 2 and 3 if the condition that something be zero is replaced by the same thing being, say, π.

4.3 Exercises

1. Which of the following collections of vectors in $\mathbb{R}^3$ are linear subspaces?

(1) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0\}$.
(2) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 0\}$.
(3) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 = 0\}$.
(4) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 = 1\}$.
(5) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 \geq 0\}$.

2. Determine the subspace of $\mathbb{R}^3$ that is the linear span of the three vectors $(1, 0, 1), (0, 1, 0), (0, 1, 1)$.

3. Repeat Exercise 2 for $(1, 0, 0), (0, 1, 0), (1, 1, 1)$.

4. Let $\mathcal{U} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = a\right\}$ for $a \in \mathbb{R}$ fixed. Show that $\mathcal{U}$ is a vector subspace if and only if $a = 0$.

5. Show that $\mathcal{P}_s(\mathbb{R})$ is a linear subspace of $\mathcal{P}_r(\mathbb{R})$ whenever $s \leq r$.

6. Show that $\mathcal{P}_k(\mathbb{R})$ is always a subspace of $\mathcal{P}(\mathbb{R})$.

7. What is the span of $\{1 + x, 1 - x\}$ in $\mathcal{P}(\mathbb{R})$?

8. What is the span of $\left\{1, x^2, x^4\right\}$ in $\mathcal{P}_4(\mathbb{R})$?

The vectors in $\mathcal{P}_n(\mathbb{R})$ are polynomials $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ of degree less than or equal to n, that is, $m \leq n$.
The vector addition is to be ordinary addition of polynomials
The scalar multiplication is the ordinary product of a polynomial by a number.

9. Find a vector that spans the subspace $2x - 3y = 0$ of $\mathbb{R}^2$.

10. Find a pair of vectors that span the subspace $x + y - 2z = 0$ of $\mathbb{R}^3$.

11. Suppose $\mathcal{S}, \mathcal{T}$ are subspaces of $\mathcal{V}$ and $\mathcal{S} \cap \mathcal{T} = \mathbf{0}$. Show that every vector in $\mathcal{S} + \mathcal{T}$ can be written *uniquely* in the form $\mathbf{A} + \mathbf{B}$, with $\mathbf{A} \in \mathcal{S}$, $\mathbf{B} \in \mathcal{T}$. Construct an example to show that this is false if $\mathcal{S} \cap \mathcal{T} \neq \mathbf{0}$.

12. Show that any nonzero vector spans $\mathbb{R}^1$.

13. Show that the two sets of vectors

$$\{\mathbf{A} = (1, 1, 0), \mathbf{B} = (0, 0, 1)\}$$

and

$$\{\mathbf{C} = (1, 1, 1), \mathbf{D} = (-1, -1, 1)\}$$

span the same subspace of $\mathbb{R}^3$.

14. Let $\mathcal{V}$ be the set of pairs of numbers $\mathbf{A} = (a_1, a_2)$. If $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ define

$$\mathbf{A} + \mathbf{B} = (a_1 + b_1, a_2 + b_2),$$

where $\mathbf{B} = (b_1, b_2)$. If a is a number define

$$a\mathbf{A} = (aa_1, 0).$$

Is $\mathcal{V}$ a vector space? Why?

15. Let ℓ_∞ be the vector space of bounded sequences (see Chapter 3, Exercise 16). Which of the following collections of vectors in ℓ_∞ are linear subspaces?

(1) $\left\{(a_0, a_1, \ldots, a_n, \ldots) \mid |a_i| < 5 \ \forall\, i \in \mathbb{N}_0\right\}$.
(2) $\left\{(a_0, a_1, \ldots, a_n, \ldots) \mid a_i = 0 \ \forall\, i > i_0,\ i_0 \in \mathbb{N}_0 \text{ fixed}\right\}$.
(3) $\left\{(a_0, a_1, \ldots, a_n, \ldots) \mid a_i = a_j \text{ for all } i \text{ and } j\right\}$.
(4) $\left\{(a_0, a_1, \ldots, a_n, \ldots) \mid a_i \in \mathbb{Z} \ \forall\, i \in \mathbb{N}_0\right\}$.

16. Suppose $\mathcal{V}$ is a vector space and E, F are subsets of $\mathcal{V}$. Show

(1) $E \subseteq F \Rightarrow \mathcal{L}(E) \subseteq \mathcal{L}(F)$.
(2) $\mathcal{L}(E \cup F) = \mathcal{L}(E) + \mathcal{L}(F)$.
(3) $\mathcal{L}(E \cap F) \subseteq \mathcal{L}(E) \cap \mathcal{L}(F)$.

17. Let $\mathcal{V}$ be a vector space and $E, F \subseteq \mathcal{V}$. Suppose $\mathcal{L}(E) \subseteq \mathcal{L}(F)$. Is it true that $E \subseteq F$?

18. Suppose that $\mathcal{V}$ is a vector space and $E \subseteq \mathcal{V}$. If $\mathcal{U}$ is a subspace containing E, show that $\mathcal{U}$ contains $\mathcal{L}(E)$.

$\mathcal{P}(\mathbb{R})$ is defined similarly, but we remove the restriction on the degree of the polynomials.

19. Suppose $\mathcal{V}$ is a vector space and $E \subseteq \mathcal{V}$. Show that $\mathcal{L}(E) = \cap\{\mathcal{U} \mid \mathcal{U}$ is a subspace of $\mathcal{V}$ and $\mathcal{U}$ contains $E\}$.

20. For any subspace $\mathcal{U}$ of $\mathcal{V}$ show that $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

21. Let $\mathcal{S}$ and $\mathcal{T}$ be linear subspaces of the vector space $\mathcal{V}$. Show that $\mathcal{S} \cup \mathcal{T} = \mathcal{V}$ if and only if $\mathcal{S}$ or $\mathcal{T}$, or both, is equal to $\mathcal{V}$.

22. Find all the linear subspaces of $\mathbb{R}^2$.

23. Find all the linear subspaces of $\mathbb{R}^3$.

24. Show that a subset E of a vector space $\mathcal{V}$ that does not contain $\mathbf{0}$ is not a subspace of $\mathcal{V}$.

25. Let E be the subset of $\mathbb{R}^2$ defined by $E = \left\{(x, y) \mid x \geq 0,\ y \in \mathbb{R}\right\}$. Is E a subspace of $\mathbb{R}^2$?

26. Let $E = \left\{(x, 2x + 1) \mid x \in \mathbb{R}\right\}$, so E is a subset of $\mathbb{R}^2$. Is E a subspace of $\mathbb{R}^2$?

27. (1) Let $E = \left\{(2a, a) \mid a \in \mathbb{R}\right\}$. Is E a subspace of $\mathbb{R}^2$?
(2) Let $B = \left\{(b, b) \mid b \in \mathbb{R}\right\}$. Is B a subspace of $\mathbb{R}^2$?
(3) What is $E \cap B$?
(4) Is $E \cup B$ a subspace of $\mathbb{R}^2$?
(5) What is $E + B$?

28. Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of $\mathcal{V}$. Prove:
(1) $\mathcal{S} + \mathcal{T} = \mathcal{L}(\mathcal{S} \cup \mathcal{T})$.
(2) $\mathcal{S} \cap (\mathcal{S} + \mathcal{T}) = \mathcal{S}$.
(3) $\mathcal{S} + \mathcal{T} = \mathcal{T} + \mathcal{S}$.
(4) If $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S} + \mathcal{T} = \mathcal{T}$.

$\mathcal{L}(E)$ denotes the linear span of E.
Namely. $\mathcal{L}(E) = \mathrm{Span}(E)$.

29. Let $\mathcal{V}$ be a vector space, $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$. Suppose $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$. Show that $\mathcal{L}(\mathbf{A}, \mathbf{B}) = \mathcal{L}(\mathbf{B}, \mathbf{C})$.

30. Let $\mathcal{V}$ be a vector space and $\mathcal{W}$ a subspace of $\mathcal{V}$. Show that

$$\left\{\mathbf{A} \in \mathcal{V} \mid \mathbf{A} \notin \mathcal{W}\right\}$$

is not a vector subspace of $\mathcal{V}$.

31. Let $\mathcal{V}$ be a vector space, $\mathcal{W}$ a subspace of $\mathcal{V}$, and $\mathbf{A}, \mathbf{B} \in \mathcal{V}$. Assume that

$$\mathbf{A} \notin \mathcal{W} \text{ but } \mathbf{A} \in \mathcal{L}(\mathcal{W} \cup \{\mathbf{B}\}).$$

Show that $\mathbf{B} \in \mathcal{L}(\mathcal{W} \cup \{\mathbf{A}\})$.

32. Let $\mathcal{S}, \mathcal{T}, \mathcal{U}$ be subspaces of a vector space $\mathcal{V}$. Is it always true that

$$\mathcal{S} \cap (\mathcal{T} + \mathcal{U}) = (\mathcal{S} \cap \mathcal{T}) + (\mathcal{S} \cap \mathcal{U})?$$

33. Can it happen for two subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{V}$ that $\mathcal{S} \cap \mathcal{T} = \varnothing$?

34. Can you find two vectors $\mathbf{A}, \mathbf{B} \in \mathbb{R}^3$ such that $\mathcal{L}(\mathbf{A}, \mathbf{B}) = \mathbb{R}^3$?

35. Let $\mathcal{V}$ be a vector space, $\mathcal{W}$ a subspace of $\mathcal{V}$, S a set, and $s \in S$. Show that

$$\mathcal{U} = \left\{f \in \mathrm{Fun}(S, \mathcal{V}) \mid f(s) \in \mathcal{W}\right\}$$

is a subspace of $\mathrm{Fun}(S, \mathcal{V})$.

36. Let S be a set and $T \subseteq S$. Is it true that

$$\mathrm{Fun}(S) = \mathrm{Fun}(S, T) + \mathrm{Fun}(S, S - T)?$$

37. Let $\mathcal{V}$ be a vector space and S a set. Let $\mathrm{Fun}(S, \mathcal{V})$ be defined as in Chapter 3 Exercise 10
(1) If $\mathcal{V}$ is a subspace of $\mathcal{W}$, show that $\mathrm{Fun}(S, \mathcal{V})$ is a subspace of $\mathrm{Fun}(S, \mathcal{W})$.
(2) If $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathcal{V}$, show that

$$\mathrm{Fun}(S, \mathcal{S}) + \mathrm{Fun}(S, \mathcal{T}) = \mathrm{Fun}(S, \mathcal{S} + \mathcal{T}).$$

EXAMPLE 1: Let $E = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$ be vectors in $\mathbb{R}^3$. Find a linearly independent set F that is a subset of E such that $\mathcal{L}(F) = \mathcal{L}(E)$.

SOLUTION: The proof of Theorem 5.2.4 suggests that we look for a vector in E linearly dependent on the remaining vectors of E and throw it away. If that doesn't lead to a linearly independent set with the same span, do it again, etc. Observe that

$$(1, 1, 1) = 1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1),$$

so that setting $F = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ we have $\mathcal{L}(F) = \mathcal{L}(E)$. The set F, however, is a linearly independent set of vectors and so we are done.

ALTERNATIVE SOLUTION: Proceeding as in the previous solution, we note that $(1, 0, 0) = 1(1, 1, 1) + (-1)(0, 1, 0) + (-1)(0, 0, 1)$, and setting $H = \{(1, 1, 1), (0, 1, 0), (0, 0, 1)\}$ we have, $\mathcal{L}(H) = \mathcal{L}(E)$. The set H, however, is a linearly independent set of vectors and so we done.

MORAL: If E is a finite set of vectors in $\mathcal{V}$, then there need not exist a *unique* subset $F \subseteq E$ with F linearly independent and $\mathcal{L}(F) = \mathcal{L}(E)$.

EXAMPLE 2: Let S be a *finite* set and $X = \left\{\chi_s \mid s \in S\right\} \subseteq \mathrm{Fun}(S)$. These characteristic functions span $\mathrm{Fun}(S)$ because for any $f \in \mathrm{Fun}(S)$ we have the equality

$$f = \sum_{s \in S} f(s)\chi_s.$$

This is seen by evaluating the right hand side at an arbitrary element $t \in S$, giving

$$\left[\sum_{s \in S} f(s)\chi_s\right](t) = \sum_{s \in S} f(s) \cdot \chi_s(t) = \left[\sum_{s \neq t} f(s) \cdot 0\right] + f(t) = f(t)$$

as required.

REMARK: Note that none of the functions χ_s for $s \in S$ can be left out and still leave us with a set of functions that span $\mathrm{Fun}(S)$, for the only way to write χ_s as a linear combination of characteristic functions is as

$$\chi_s = \sum_{t \in S} \delta_{s,t}\chi_t = 1 \cdot \chi_s.$$

REMARK: Example 2 cannot be extended to infinite sets. For if S is an infinite set then the function

$$f : S \to \mathbb{R}$$

defined by $f(s) = 1$, $s \in S$ is not a *(finite)* linear combination of characteristic functions.

5.3 Exercises

1. Which of the following sets of vectors in $\mathbb{R}^3$ are linearly dependent and which are linearly independent?

$E = \{(1, 1, 1), (0, 1, 0), (1, 0, 1)\}$.
$F = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$.
$G = \{(1, 1, 1), (1, 1, 0), (1, 0, 1)\}$.
$H = \{(1, 0, 0), (0, 1, 0), (1, 1, 1)\}$.
$K = \{(1, 1, 1), (0, 1, 0), (0, 0, 1)\}$.

2. Which of the following sets of vectors in $\mathcal{P}_2(\mathbb{R})$ are linearly dependent and which are linearly independent?

$E = \left\{1, x, x^2\right\}$.
$F = \left\{1 + x, 1 - x, x^2, 1\right\}$.
$G = \left\{x^2 - 1, x + 1, x^2 - x, x^2 + x\right\}$.
$H = \left\{x - x^2, x^2 - x\right\}$.
$K = \left\{1, 1 - x, 1 - x^2\right\}$.

3. Show that the set of vectors in $\mathbb{R}^3$

$$E = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$$

is linearly dependent but that any set of three of them is linearly independent.

4. Let ℓ_∞ be the vector space of bounded sequences of numbers (see Chapter 3 Exercise 16). Which of the following collections of vectors in ℓ_∞ are linearly independent?

(1) $\{(1, 1, \ldots, 1, \ldots), (2, 1, 1, \ldots, 1, \ldots), (0, 1, 1, \ldots, 1, \ldots)\}$.
(2) $\{(1, 0, \ldots, 0, \ldots), (0, 1, 0, , \ldots, 0, \ldots), (0, 0, \ldots, 0, 1, 0 \ldots), \ldots\}$.
(3) $\{(1, , 1, 1, 0, \ldots, 0, \ldots), (1, 1, 0, \ldots, 0, \ldots), (1, 0, 1, 0, \ldots, 0, \ldots)\}$.
(4) $\{(1, 0, 0, \ldots, 0, \ldots), (1, 1, 0, , \ldots, 0, \ldots), (1, 1, \ldots, 1, 0, \ldots, 0), \ldots, \}$.
(5) $\{(1, 1/2, \ldots, 1/n, \ldots), (0, 1/2, 1/3, \ldots, 1/n, \ldots),$
$(0, 0, 1/3, \ldots, 1/n, \ldots), \ldots, (0, , \ldots, 0, 1/n, \ldots), \ldots, \}$.

5. Let $\mathcal{U}$ be the subspace of $\mathbb{R}^5$ spanned by the vectors

$$E = \{(1, 1, 0, 0, 1),\ (1, 1, 0, 1, 1),\ (0, 1, 1, 1, 1),\ (2, 1, 1, 0, 1)\}.$$

Find a linearly independent subset F of E with $\mathcal{L}(F) = \mathcal{U}$.

6. Let $\mathcal{U}$ be the subspace of $\mathcal{P}_3(\mathbb{R})$ spanned by

$$E = \left\{x^3,\ x^3 - x^2,\ x^3 + x^2,\ x^3 - 1\right\},$$

find a linearly independent subset F of E spanning $\mathcal{U}$. (**HINT:** Try to use Theorem 5.2.4.)

7. For each of the linearly dependent sets S in Exercise 4 use Theorem 5.2.4 to find a linearly independent subset of vectors with the same span as S.

8. Under what conditions on the numbers a and b are the vectors $(1, a)$, $(1, b)$ linearly independent in $\mathbb{R}^2$.

9. Suppose that E and F are sets of vectors in $\mathcal{V}$ with $E \subset F$. Prove that if E is linearly dependent, then so is F.

10. Suppose that E and F are sets of vectors in $\mathcal{V}$ with $E \subset F$. Prove that if F is linearly independent then so is E.

11. Show that functions e^x and e^{2x} form a set of linearly independent vectors in $\mathcal{C}(-\infty, \infty)$.

12. Show that

$$\left\{\cos x,\ \sin x,\ \sin\left(x + \frac{\pi}{2}\right)\right\}$$

is a set of linearly dependent vectors in $\mathcal{C}(-\infty, \infty)$.

The vectors of $C(-\infty, \infty)$ are the continuous functions defined on $(-\infty, \infty)$. Addition of vectors is to be addition of functions. $(f+g)(x) = f(x)+g(x)$ Scalar multiplication is the ordinary product of a function by a number $(c \cdot f)(x) = c \cdot f(x)$

13. Is the pair of complex numbers $\alpha + \beta i$, $\alpha - \beta i$ a set of linearly independent vectors in $\mathbb{C}$, where α, β are any nonzero real numbers?

14. Show that the set of polynomials $E = \left\{x^2, 1 + x^2\right\}$ is a set of linearly independent vectors in $\mathcal{P}_2(\mathbb{R})$. What is the space spanned by E? Is $\mathcal{L}(E) = \mathcal{P}_2(\mathbb{R})$? If not, find a vector in $\mathcal{P}_2(\mathbb{R})$ that does not belong to $\mathcal{L}(E)$, and show that together with E they form a set of linearly independent vectors in $\mathcal{P}_2(\mathbb{R})$.

15. Let E be a set of vectors in the vector space $\mathcal{V}$. Show that E is linearly independent if and only if for every proper subset E' of E, $\mathcal{L}(E') \neq \mathcal{L}(E)$. Equivalently, show that E is linearly dependent if and only if there exists a proper subset E' of E such that $\mathcal{L}(E') = \mathcal{L}(E)$.

16. Let $S = \{x_1, x_2, x_3\}$ and consider the functions $f, g, h \in \mathrm{Fun}(S)$ defined by

$$\begin{aligned} f(x_1) &= 0, & f(x_2) &= 1, & f(x_3) &= 1, \\ g(x_1) &= 1, & g(x_2) &= 0, & g(x_3) &= 1, \\ h(x_1) &= 1, & h(x_2) &= 1, & h(x_3) &= 0. \end{aligned}$$

Does $\{f, g, h\}$ form a linearly independent set?

17. Let E', E'' be linearly independent sets of vectors in $\mathcal{V}$. Show that:
(1) $E' \cap E''$ is linearly independent.
(2) $E' \cup E''$ is linearly independent if and only if $\mathcal{L}(E') \cap \mathcal{L}(E'') = \{0\}$.

18. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$ be linearly independent vectors. Show that the vectors $\mathbf{A} + \mathbf{B}$, $\mathbf{B} + \mathbf{C}$, $\mathbf{C} + \mathbf{A}$ are linearly independent.

19. Let $\mathcal{V}$ be a vector space and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{V}$. Show that the vectors

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D},\quad 2\mathbf{A} + 2\mathbf{B} + \mathbf{C} \quad \mathbf{D},\quad \mathbf{A} \quad \mathbf{B} + \mathbf{C},\quad \mathbf{A} \quad \mathbf{C} + \mathbf{D}$$

are linearly dependent. Find a maximal linearly independent subset of these four vectors.

20. Let $\mathcal{S}, \mathcal{T}$ be subspaces of a vector space $\mathcal{V}$, and $\mathbf{S} \in \mathcal{S}$, $\mathbf{T} \in \mathcal{T}$, $\mathbf{S} \neq \mathbf{0}$, $\mathbf{T} \neq \mathbf{0}$. Assume $\mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$. Show that $\{\mathbf{S}, \mathbf{T}\}$ is a linearly independent set of vectors.

21. Show that any four vectors in $\mathbb{R}^3$ must be linearly dependent.

22. Let $\mathcal{S}, \mathcal{T}$ be subspaces of $\mathcal{V}$ and S a set. Suppose $f \in \mathrm{Fun}(S, \mathcal{S})$, $g \in \mathrm{Fun}(S, \mathcal{T})$, and $\mathcal{S} \cap \mathcal{T} = \mathbf{0}$. Show that regarded as vectors in $\mathrm{Fun}(S, \mathcal{V})$, the vectors f and g are linearly independent.

23. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$. If $\mathbf{C} \notin \mathcal{L}(\mathbf{A}, \mathbf{B})$, show that $\mathbf{A}$ and $\mathbf{B}$ are linearly independent if and only if $\mathbf{A} + \mathbf{C}$ and $\mathbf{B} + \mathbf{C}$ are linearly independent.

$\mathcal{L}(E) = \mathrm{Span}(E)$

24. Let $f_1, \ldots, f_k \in \mathcal{P}(\mathbb{R})$ be polynomials of positive degree. Suppose $f_1, \ldots, f_k$ are linearly independent. Does it follow that their derivatives $df_1/dx, \ldots, df_k/dx$ are linearly independent?

25. Let S be a set. A **permutation** σ of S is a bijective (i.e., a one to one and onto) mapping $\sigma : S \to S$. For a permutation σ of S and $f \in \mathrm{Fun}(S)$ define

$$f_\sigma : S \to \mathbb{R} \text{ by } f_\sigma(s) = f(\sigma(s)).$$

Suppose $f_1, \ldots, f_n \in \mathrm{Fun}(S)$ are linearly independent. Show for any permutation σ that $(f_1)_\sigma, \ldots, (f_n)_\sigma$ are also linearly independent.

26. If $f_1, \ldots, f_n \in \mathcal{P}_m(\mathbb{R})$ are linearly dependent then so are their derivatives $df_1/dx, \ldots, df_n/dx$.

EXAMPLE 6: For what values of r are the vectors $(r, 1, 1), (1, r, 1), (1, 1, r)$ a basis for $\mathbb{R}^3$?

SOLUTION: Since $\dim(\mathbb{R}^3) = 3$, it will suffice to determine when the vectors in question are linearly independent. So suppose that

$$a(r, 1, 1) + b(1, r, 1) + c(1, 1, r) = 0$$

is a linear relation. Then

$$(ar + b + c, a + br + c, a + b + cr) = \mathbf{0} = (0, 0, 0),$$

so taking coordinates gives

$$\begin{aligned} ar + b + c &= 0, \\ a + br + c &= 0, \\ a + b + cr &= 0. \end{aligned}$$

From the last equation we get

$$(*) \qquad cr = -(a + b).$$

Multiplying the second equation by r and substituting gives

$$\begin{aligned} 0 &= ar + br^2 - (a + b) = a(r - 1) + b(r^2 - 1) \\ &= (r - 1)[a + b(r + 1)]. \end{aligned}$$

So if $r - 1 \neq 0$, we may divide by $r - 1$ and solve for a to get

$$(**) \qquad a = -(r + 1)b.$$

Multiplying the first equation by r and substituting in $(*)$ and $(**)$, we obtain

$$\begin{aligned} 0 &= [-(r + 1)b]r^2 + br - (a + b) \\ &= -r^2(r + 1)b + rb - [-(r + 1)b + b] \\ &= [-r^2(r + 1) + r + (r + 1) - 1]b \\ &= (-r^3 - r^2 + r + r)b = -(r^3 + r^2 - 2r)b \\ &= -r(r^2 + r - 2)b = -r(r + 2)(r - 1)b. \end{aligned}$$

So assuming $r \neq 0, -2, 1$ we may divide by $-r(r + 2)(r - 1)$ and conclude that $b = 0$, whence $a = 0$ by $(**)$ and $c = 0$ by $(*)$. Therefore, the vectors are independent when $r \neq 0, 1, -2$. For $r = 0$, the original system simplifies to

$$\begin{aligned} 0 &= b + c, \\ 0 &= a + c, \\ 0 &= a + b. \end{aligned}$$

Substituting the first equation into the second gives

$$0 = a - b$$

and adding to the third gives

$$0 = 2a,$$

giving $a = 0$ and $b = 0$, $c = 0$, so again the vectors are independent. For $r = 1$, the three vectors are the same vector three times, so they cannot be a basis for $\mathbb{R}^3$. And finally, for $r = -2$, there is the linear relation

$$(-2, 1, 1) + (1, -2, 1) + (1, 1, -2) = (0, 0, 0),$$

so the vectors are not linearly dependent. In conclusion, the vectors $(r, 1, 1), (1, r, 1), (1, 1, r)$ are a basis for $\mathbb{R}^3$ if and only if $r \neq 1, -2$.

6.4 Exercises

1. Which of the following sets of vectors are bases for $\mathbb{R}^3$?

$$\begin{aligned} E &= \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}. \\ F &= \{(1, 0, 1), (0, 1, 0), (1, 1, 1)\}. \\ G &= \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}. \\ H &= \{(1, -1, 0), (1, 0, -1), (-1, 0, 1)\}. \end{aligned}$$

2. Which of the following sets of vectors are bases for $\mathcal{P}_2(\mathbb{R})$?

$$\begin{aligned} E &= \{1, 1 + x, 1 + x + x^2\}. \\ F &= \{1, (x - 1), (x - 1)(x - 2)\}. \\ G &= \{x^2 + x + 1, x^2 - x + 1, x^2 - x - 1\}. \\ H &= \{x^2 + x + x^2 + 1, x + 1\}. \end{aligned}$$

3. Find a basis for each of the following subspaces of $\mathbb{R}^3$.

$$\begin{aligned} \mathcal{U} &= \{(x, y, z) \mid x - z = 0\}. \\ \mathcal{S} &= \{(x, y, z) \mid x + y + z = 0\}. \\ \mathcal{T} &= \{(x, y, z) \mid x - y + z = 0\}. \\ \mathcal{W} &= \{(x, y, z) \mid x = 0 \text{ and } y + z = 0\}. \end{aligned}$$

4. Find a basis for each of the folowing the subspaces of $\mathcal{P}_3(\mathbb{R})$:

$$\mathcal{V} = \left\{ p(x) \mid p(0) = 0 \right\}.$$

$$\mathcal{S} = \left\{ p(x) \mid \frac{d}{dx} p(x) = 0 \right\}.$$

$$\mathcal{T} = \left\{ p(x) \mid p(x) = a_0 + a_1 x + a_3 x^3 \right\}.$$

$$\mathcal{W} = \left\{ p(x) \mid p(-x) = p(x) \right\}.$$

5. Calculate the dimension of each of the subspaces in the preceding two exercises.

6. Let ℓ_∞ denote the vector space of bounded sequences (see Chapter 3 Exercise 16). Which of the following sets of vectors are bases for ℓ_∞?

$$E = \{(1, 0, \ldots, 0, \ldots), (0, 1, 0, \ldots, 0, \ldots), \ldots, (0, , \ldots, 0, 1, 0, \ldots), \ldots\}.$$

$$F = \{(1, 1, , \ldots, 1, \ldots), (0, 1, \ldots, 1, \ldots), \ldots, (0, 0, \ldots, 0, 1, 1, \ldots), \ldots\}.$$

$$F = \{(1, 1, \ldots, 1, \ldots), (2, 1, \ldots, 1, \ldots), \ldots, (n, 1, \ldots, 1, \ldots), \ldots\}.$$

$$G = \{(1, 0, \ldots, 0, \ldots), (0, 1/2, 0, \ldots, 0, \ldots), \ldots, (0, \ldots, 1/n, 0, \ldots), \ldots\}.$$

7. For each of the following subsets of ℓ_∞ show that the subset is a linear subspace and find a basis.

$$\mathcal{A} = \left\{ (a_0, a_1, \ldots, a_n, \ldots) \mid a_i = a_j \ \forall\, i, j \in \mathbb{N}_0 \right\},$$

$$\mathcal{S}_{n_0} = \left\{ (a_0, a_1, \ldots, a_n, \ldots) \mid a_i = 0 \ \forall\, i > n_0 \right\},$$ where $n_0 \in \mathbb{N}_0$ is a fixed nonnegative integer.

8. Suppose that $\mathcal{V}$ is a finite-dimensional vector space and $\mathcal{S}$ is a linear subspace of $\mathcal{V}$. Show that there exists a linear subspace $\mathcal{T}$ of $\mathcal{V}$ such that $\mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$ and $\mathcal{S} + \mathcal{T} = \mathcal{V}$. (Hint: Study the proof of the basis extension theorem.)

9. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are subspaces of a finite-dimensional vector space $\mathcal{V}$, and $\mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$. Show that

$$\dim(\mathcal{S} + \mathcal{T}) = \dim(\mathcal{S}) + \dim(\mathcal{T}).$$

10. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are subspaces of a finite-dimensional vector space $\mathcal{V}$. Show that

$$\dim(\mathcal{S} + \mathcal{T}) = \dim(\mathcal{S}) + \dim(\mathcal{T}) - \dim(\mathcal{S} \cap \mathcal{T}).$$

(**Hint:** Choose a basis for $\mathcal{S} \cap \mathcal{T}$, extend it to a basis for $\mathcal{S}$, and extend it to a basis for $\mathcal{T}$. Show that the result spans $\mathcal{V}$ and count the number of vectors that occur twice.)

11. Suppose that $\mathcal{S}_1, \ldots, \mathcal{S}_k$ are subspaces of a finite-dimensional vector space $\mathcal{V}$. Find a formula for the dimension of $\mathcal{S}_1 + \cdots + \mathcal{S}_k$. (**Hint:** Try the case $k = 3$.)

12. What is the dimension of $\mathbb{C}$ as a vector space over $\mathbb{R}$? What is the dimension of $\mathbb{C}$ as a vector space over $\mathbb{C}$?

13. Let $\mathcal{S} = \{(a, 0, 0)\}$ and $\mathcal{T} = \{(0, b, b)\}$, where a and b are real numbers. Show $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathbb{R}^2$. Find a basis for $\mathcal{S} + \mathcal{T}$.

14. Let $\mathcal{S}$ be the subspace of $\mathbb{R}^3$ given by

$$\mathcal{S} = \left\{ (x, y, z) \mid y - z = 0 \right\}.$$

Find a subspace $\mathcal{T}$ of $\mathbb{R}^3$ such that $\mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$ and $\mathcal{S} + \mathcal{T} = \mathbb{R}^3$.

15. Under what conditions on the number a will the vectors

$$(a, 1, 0), (1, a, 1), (0, 1, a)$$

be a basis for $\mathbb{R}^3$?

16. Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be a basis for $\mathbb{R}^n$. Set

$$\mathcal{S} = \left\{ \sum_{i=1}^{n} a_i \mathbf{A}_i \mid a_1, \ldots, a_n \in \mathbb{R} \text{ and } a_2 = \cdots = a_n \right\},$$

$$\mathcal{T} = \left\{ \sum_{i=1}^{n} a_i \mathbf{A}_i \mid a_1, \ldots, a_n \in \mathbb{R} \text{ and } \sum_{i=1}^{n} a_i = 0 \right\}.$$

Show that $\mathcal{S}$ and $\mathcal{T}$ are vector subspaces and find bases for $\mathcal{S}, \mathcal{T}, \mathcal{S} \cap \mathcal{T}$, and $\mathcal{S} + \mathcal{T}$.

17. Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be vectors in $\mathcal{V}$. Suppose that $n = \dim(\mathcal{V})$. Show that $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are linearly independent if and only if $\mathcal{L}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ has dimension n.

18. The equation $y = 3x$ defines a straight line in the plane. Show that if $\mathbf{A}, \mathbf{B}$ are points on this line then $\vec{\mathbf{A}}, \vec{\mathbf{B}}$ are linearly dependent vectors.

19. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be four distinct points on a plane Π. Show that the vectors $\vec{\mathbf{AB}}, \vec{\mathbf{AC}}, \vec{\mathbf{AD}}$ form a set of linearly dependent vectors.

20. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be three points in a space. Show that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not colinear (i.e., they all lie on the same line) if and only if $\left\{ \vec{\mathbf{AB}}, \vec{\mathbf{AC}} \right\}$ is a set of linearly independent vectors.

To find the dimension of $\mathcal{V}$, recall from Section 6.2 Example 5 that $\dim(\mathrm{Fun}(S)) = 3$. Moreover, $\mathcal{V} \neq \mathrm{Fun}(S)$ because $\chi_u \notin \mathcal{V}$. Therefore, $\dim(\mathcal{V}) < 3$; i.e., $\dim(\mathcal{V}) = 0, 1$, or 2. A little experimentation shows that the two vectors $\chi_u - \chi_w, \chi_v$ belong to $\mathcal{V}$. They are also linearly independent, for if

$$a(\chi_u - \chi_w) + b\chi_v = \mathbf{0}$$

is a linear relation, then

$$a\chi_u + b\chi_v - a\chi_w = \mathbf{0},$$

and evaluating at u and v gives

$$0 = a\chi_u(u) + b\chi_v(u) - a\chi_w(u) = a,$$
$$0 = a\chi_u(v) + b\chi_v(v) - a\chi_w(v) = b,$$

so $a = 0$, $b = 0$, as required. Therefore, $\dim(\mathcal{V}) \geq 2$, which since $\dim(\mathcal{V}) < 2$, shows that $\dim(\mathcal{V}) = 2$.

7.2 Exercises

1. Let $\mathbf{A} = (1, 0, 1)$, $\mathbf{B} = (-1, 1, 0)$, $\mathbf{C} = (0, 1, 1)$ be three vectors of $\mathbb{R}^3$. Show that $\mathcal{L}(\mathbf{A}, \mathbf{B}) = \mathcal{L}(\mathbf{B}, \mathbf{C})$.

2. $\{\mathbf{B}_1, \mathbf{B}_2\}$ is a basis for $\mathcal{L}(\mathbf{B}_1, \mathbf{B}_2)$ in Eexample 4 of this chapter. Find the coordinates (in components) of $\mathbf{A}_1$ relative to this basis $\{\mathbf{B}_1, \mathbf{B}_2\}$. Do the same problem for $\mathbf{A}_2$ and $\mathbf{A}_3$.

3. Let $\mathbf{P} = (a, b)$ be a vector in $\mathbb{R}^2$.
(1) Show that a, b are the coordinates of $\mathbf{P}$ relative to the basis $\mathbf{E}_1 = (1, 0)$, $\mathbf{E}_2 = (0, 1)$.
(2) Find the coordinates of $\mathbf{P} = (a, b)$ relative to the basis $\mathbf{E}_1 = (1, 0)$ and $\mathbf{F}_2 = (0, 2)$.
(3) Find the coordinates of $\mathbf{P}$ relative to the basis $\mathbf{G} = (1, 1)$ and $\mathbf{H} = (-1, 2)$.

4. Are the polynomials $\{x + x^3, 1 + x^2\}$ a set of linearly independent vectors of $\mathcal{P}_3(\mathbb{R})$? If so, is $\{x + x^3, 1 + x^2\}$ a basis for $\mathcal{P}_3(\mathbb{R})$? What is the dimension of $\mathcal{L}(x + x^3, 1 + x^2)$?

5. The solution space $\mathcal{S}$ of $x - 2y + 3z = 0$ is a subspace of $\mathbb{R}^3$. Show that $\dim(\mathcal{S}) = 2$. Find a basis for $\mathcal{S}$.

6. Let S be a set. A map

$$\tau : S \to S$$

is called an **involution** if

$$\tau(\tau(s)) = s, \forall\, s \in S.$$

Fun(S) is the set of all functions $f: S \to \mathbb{R}$. If $f, g \in \mathrm{Fun}(S)$ the sum $f+g: S \to \mathbb{R}$ is defined by $(f+g)(s) = f(s) + g(s)$ $\forall s \in S$. And for c a real number, $cf: S \to \mathbb{R}$ is defined by $(cf)(s) = c \cdot f(s)$ $\forall s \in S$.

Let τ be an involution on the set S and define

$$\mathrm{Fun}_+(S) = \left\{ f \in \mathrm{Fun}(S) \mid f(\tau(s)) = f(s), \forall\, s \in S \right\},$$
$$\mathrm{Fun}_-(S) = \left\{ f \in \mathrm{Fun}(S) \mid f(\tau(s)) = -f(s), \forall\, s \in S \right\}.$$

(1) Show that $\mathrm{Fun}_+(S)$ and $\mathrm{Fun}_-(S)$ are subspaces of $\mathrm{Fun}(S)$.
(2) Show that $\mathrm{Fun}_+(S) \cap \mathrm{Fun}_-(S) = \{\mathbf{0}\}$.
(3) Let $f \in \mathrm{Fun}(S)$ and define

$$f_+ : S \to \mathbb{R} \text{ by } f_+(s) = f(s) + f(\tau(s)),$$
$$f_- : S \to \mathbb{R} \text{ by } f_-(s) = f(s) - f(\tau(s)).$$

Show that $f_+ \in \mathrm{Fun}_+(S)$ and $f_- \in \mathrm{Fun}_-(S)$.
(4) Show that $\mathrm{Fun}(S) = \mathrm{Fun}_+(S) + \mathrm{Fun}_-(S)$.
(5) If S is finite show that

If S is finite, |S| is the number of elements in S.

$$\dim(\mathrm{Fun}_+(S)) + \dim(\mathrm{Fun}_-(S)) = |S|.$$

7. Let $\mathcal{V}$ be a vector space, and $\mathcal{S}, \mathcal{T}$ finite-dimensional subspaces of $\mathcal{V}$. If $\mathcal{S}$ and $\mathcal{T}$ have the same dimension and $\mathcal{S} \subseteq \mathcal{T}$, show that $\mathcal{S} = \mathcal{T}$.

8. Let $S = \{s, t, u\}$ and let $\tau : S \to S$ be the involution obtained by interchanging s and u and leaving t alone, that is

$$\tau(s) = u, \qquad \tau(t) = t, \qquad \tau(u) = s.$$

Find bases for $\mathrm{Fun}_+(S)$ and $\mathrm{Fun}_-(S)$ (see Exercise 6).

9. Find a basis for the subspaces of $\mathcal{P}_4(\mathbb{R})$ spanned by the polynomials $f \in \mathcal{P}_4(\mathbb{R})$ that satisfy the equation

$$f(z) = z^4 f\left(\frac{1}{z}\right).$$

10. Let $\mathcal{S}, \mathcal{T}, \mathcal{U}$ be vector subspaces of the vector space $\mathcal{V}$. Form the nine subspaces

$$\begin{array}{lll} \mathcal{S}, & \mathcal{S} \cap \mathcal{T}, & \mathcal{S} \cap \mathcal{U}, \\ \mathcal{S} + \mathcal{T}, & \mathcal{S} \cap (\mathcal{T} + \mathcal{U}), & \mathcal{S} + (\mathcal{T} \cap \mathcal{U}), \\ \mathcal{S} + \mathcal{T} + \mathcal{U}, & (\mathcal{S} \cap \mathcal{T}) + (\mathcal{S} \cap \mathcal{U}), & (\mathcal{S} + \mathcal{T}) \cap (\mathcal{S} + \mathcal{U}). \end{array}$$

Show that for $\mathcal{X}, \mathcal{Y}$ any two of these nine subspaces, either $\mathcal{X} \subseteq \mathcal{Y}$ or $\mathcal{Y} \subseteq \mathcal{X}$. Find an example with the maximum number of distinct subspaces among the nine.

PROOF: The polynomial

$$P_f(t) = \sum_{i=0}^{n} f(t_i)\Phi_i(t)$$

has degree n and agrees with $f(t)$ at the points $t_0, t_1, \ldots, t_n$. If $Q_f(t)$ is any other polynomial of degree n with this property, then, since $\Phi_0, \Phi_1, \ldots, \Phi_n$ is a basis for $\mathcal{P}(\mathbb{R})$ (Proposition 9.2.1), it follows that

$$Q_f = \sum_{j=0}^{n} a_j\Phi_j$$

for certain uniquely defined numbers $a_0, a_1, \ldots, a_n$. Evaluating this equation at t_i yields

$$f(t_i) = Q_f(t_i) = \sum_{j=0}^{n} a_j\Phi_j(t_i) = a_i,$$

and hence $Q_f = P_f$, as required. □

9.3 Exercises

1. Let $k(x)$ be a *fixed* polynomial. Define functions

$$\mathbf{M}_{k(x)} : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R}),$$
$$\mathbf{L}_{k(x)} : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R}),$$

by

$$\mathbf{M}_{k(x)}(p(x)) = k(x)p(x),$$
$$\mathbf{L}_{k(x)}(p(x)) = \int_0^x p(t)k(t)dt.$$

If $k(x) = 1$ for all $x \in \mathbb{R}$, we drop the subscript and write $\mathbf{M}$ and $\mathbf{L}$.

(1) Show that $\mathbf{M}_{k(x)}$ and $\mathbf{L}_{k(x)}$ are linear transformations for all polynomials $k(x)$. Calculate

$$\mathbf{M}_{x^2+2x^3}(x + x^4) \quad \text{and} \quad \mathbf{L}_{x+x^2}(1 + x^3).$$

(2) Calculate $\mathbf{D} \cdot \mathbf{M}_x - \mathbf{M}_x \cdot \mathbf{D} : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$, where $\mathbf{D}$ is defined in Example 2 of Section 8.1. That is find a *formula* for

$$(\mathbf{D} \cdot \mathbf{M}_x - \mathbf{M}_x \cdot \mathbf{D})(p(x)).$$

(3) Show that $\mathbf{L} \cdot \mathbf{M}_{k(x)} = \mathbf{L}_{k(x)} : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$.

2. Show that $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{T}(x, y) = (ax + by, cx + dy),$$

a, b, c, d fixed real numbers, is a linear transformation and that $\mathbf{T}$ is an isomorphism if and only if $ad \neq bc$.

3. Show that $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{T}(x, y, z) = (a_1x + a_2y + a_3z, b_1x + b_2y + b_3z, c_1x + c_2y + c_3z)$$

is a linear transformation. Any linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ takes this form, where a_i, b_j, c_k are scalars.

4. Let $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $\mathbf{T}(x, y) = (x - y, 2x + y)$ and let $\mathbf{S} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $\mathbf{S}(x, y) = (y - 2x, x + y)$.

(1) Find $\mathbf{T} \cdot \mathbf{S}(1, 0)$, $(\mathbf{T} + \mathbf{S})(1, 0)$, $\mathbf{S} \cdot \mathbf{T}(1, 0)$, $(2\mathbf{T} - \mathbf{S})(1, 0)$, $\mathbf{S}^2(1, 0)$, and $\mathbf{T}^2(1, 0)$.

(2) Find $\mathbf{T} \cdot \mathbf{S}(x, y)$ and $\mathbf{S} \cdot \mathbf{T}(x, y)$.

(3) What are the vectors (x, y) satisfying $\mathbf{T}(x, y) = (1, 0)$?

5. Let $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{T}(x, y) = (x, x)$. What are the kernel of $\mathbf{T}$ and the image of $\mathbf{T}$?

6. Let $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{T}(x, y, z) = (x + y, y + z, z + x)$. Is $\mathbf{T}$ an isomorphism? Find $\mathbf{T}(1, 0, 0)$, $\mathbf{T}(0, 1, 0)$ and $\mathbf{T}(0, 0, 1)$.

7. Let $\mathbf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation satisfying the condition

$$\mathbf{T}(x, y, z) = (0, 0, 0) \quad \text{whenever} \quad 2x - y + z = 0.$$

Let $\mathbf{T}(0, 0, 1) = (1, 2, 3)$. Find $\mathbf{T}(1, 0, 0)$ and $\mathbf{T}(0, 1, 0)$. What is $\dim(\mathrm{Im}(\mathbf{T}))$?

8. Let $\mathbf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by the formula

$$\mathbf{T}(x, y, z) = (y + z, x + z, x + y).$$

Show that $\mathbf{T}$ is an isomorphism and find an inverse for $\mathbf{T}$.

9. Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{R}$, and $\mathbf{E}_1, \ldots, \mathbf{E}_n$ a basis for $\mathcal{V}$. Define for each i, $1 \leq i \leq n$, a function

$$\mathbf{E}_i^* : \mathcal{V} \longrightarrow \mathbb{R}$$

by the formula

$$\mathbf{E}_i^*(\mathbf{A}) = a_i, \qquad 1 \leq i \leq n,$$

where

$$\mathbf{A} = a_1\mathbf{E}_1 + \cdots + a_n\mathbf{E}_n.$$

(1) Show that $\mathbf{E}_1^*, \ldots, \mathbf{E}_n^*$ are linear transformations.

(2) Find a basis for $\ker(\mathbf{E}_i^*)$.

(3) Let $\mathbf{S} : \mathcal{V} \to \mathcal{L}(V, \mathbb{R})$ be the linear extension of the map

$$\mathbf{E}_i \longrightarrow \mathbf{E}_i^*, \qquad i = 1, \ldots, n.$$

Show that $\mathbf{S}$ is an isomorphism.

(4) Compute $\dim_{\mathbb{R}} \mathcal{L}(\mathcal{V}, \mathbb{R})$.

10. Let $\mathcal{V}$ be a vector space and define $\mathcal{V}^* = \mathcal{L}(\mathcal{V}, \mathbb{R})$. Define a linear transformation

$$\mathbf{B} : \mathcal{V} \longrightarrow \mathcal{V}^{**} \qquad (= \mathcal{L}(\mathcal{V}^*, \mathbb{R}) = \mathcal{L}(\mathcal{L}(\mathcal{V}, \mathbb{R}), \mathbb{R}))$$

by

$$\mathbf{B}(\mathbf{A})(f) = f(\mathbf{A}).$$

(1) Show that $\mathbf{B}$ is a linear transformation.
(2) Show that $\ker(\mathbf{B}) = \{\mathbf{0}\}$.
(3) Show that $\mathbf{B}$ is an isomorphism if $\mathcal{V}$ is finite-dimensional.
(4) Show by an example that $\mathrm{Im}(\mathbf{B}) \neq \mathcal{V}^{**}$ when $\mathcal{V}$ is not finite-dimensional.

11. Let $E = \{\mathbf{E}_1, \ldots, \mathbf{E}_n\}$ be a basis for the finite-dimensional vector space $\mathcal{V}$. Show that the linear extension construction gives an isomorphism

$$\mathbf{L} : \mathrm{Fun}(E) \to \mathcal{V}$$

by

$$\mathbf{L}(f) = f(\mathbf{E}_1) \cdot \mathbf{E}_1 + \cdots + f(\mathbf{E}_n) \cdot \mathbf{E}_n.$$

12. Let $E = \{\mathbf{E}_1, \ldots, \mathbf{E}_n\}$ be a basis for the finite-dimensional vector space $\mathcal{V}$. Show that the map (remember, $\mathcal{V}^* = \mathcal{L}(\mathcal{V}, \mathbb{R})$ and the remarks following Theorem 8.5.6)

$$\mathbf{C} : \mathcal{V}^* \to \mathrm{Fun}(E)$$

defined by

$$\mathbf{C}(\mathbf{T})(\mathbf{E}_i) = \mathbf{T}(\mathbf{E}_i), \qquad i = 1, \ldots, n,$$

is a vectorspace isomorphism.

13. Define a function

$$\mathbf{S} : \mathcal{P}_n(\mathbb{R}) \longrightarrow \mathcal{P}_n(\mathbb{R})$$

by

$$\mathbf{S}(p(x)) = p(x + 1).$$

Determine whether $\mathbf{S}$ is a linear transformation.

14. Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces. Suppose $E = \{\mathbf{E}_1, \ldots, \mathbf{E}_n\}$ is a basis for $\mathcal{V}$ and $\{\mathbf{F}_1, \ldots, \mathbf{F}_m\}$ a basis for $\mathcal{W}$. For

each ordered pair (i, j) show that the function $\mathbf{M}_{i,j} : \mathcal{V} \longrightarrow \mathcal{W}$, defined by the formula

$$\mathbf{M}_{i,j}(\mathbf{A}) = a_j \mathbf{F}_i, \qquad \mathbf{A} = a_1\mathbf{E}_1 + \cdots + a_n\mathbf{E}_n,$$

is a linear transformation.

(1) Show that $\left\{\mathbf{M}_{i,j} \mid 1 \le i \le n,\ 1 \le j \le \right\}$ is a basis for $\mathcal{L}(V, W)$.
(2) Show that $\dim(\mathcal{L}(\mathcal{V}, \mathcal{W})) = nm$.

15. Let $\mathcal{V}$ be a vector space and suppose that $\mathbf{S}, \mathbf{T} \in \mathcal{V}^* = \mathcal{L}(\mathcal{V}, \mathbb{R})$. Define

$$\mathbf{L} : \mathcal{V} \longrightarrow \mathbb{R}^2$$

by

$$\mathbf{L}(\mathbf{A}) = (\mathbf{S}(\mathbf{A}), \mathbf{T}(\mathbf{A})).$$

(1) Show that $\mathbf{L}$ is a linear transformation.
(2) Show that $\ker(\mathbf{L}) = \ker(\mathbf{S}) \cap \ker(\mathbf{T})$.

16. Let $\mathcal{V}$ be a vector space and define (see Theorem 8.5.6)

$$\mathbf{C}_i : \underset{\substack{\shortparallel \\ \mathrm{Hom}(\mathcal{V}, \mathbb{R})}}{\mathcal{L}(\mathcal{V}, \mathbb{R})} \to \underset{\substack{\shortparallel \\ \mathrm{Hom}(\mathcal{V}, \mathbb{R}^2)}}{\mathcal{L}(\mathcal{V}, \mathbb{R}^2)}, \qquad i = 1, 2,$$

by

$$\begin{aligned} \mathbf{C}_1(\mathbf{T})(\mathbf{A}) &= (\mathbf{T}(\mathbf{A}), 0), \\ \mathbf{C}_2(\mathbf{T})(\mathbf{A}) &= (0, \mathbf{T}(\mathbf{A})), \end{aligned} \qquad \forall\, \mathbf{A} \in \mathcal{V}, \mathbf{T} \in \underset{\substack{\shortparallel \\ \mathrm{Hom}(\mathcal{V}, \mathbb{R})}}{\mathcal{L}(\mathcal{V}, \mathbb{R})}.$$

(1) Show that $\mathbf{C}_i$ is a linear transformation for $i = 1, 2$.
(2) Show that $\ker(\mathbf{C}_i) = \{\mathbf{0}\}$, $i = 1, 2$.
(3) Show that $\mathrm{Im}(\mathbf{C}_1) \cap \mathrm{Im}(\mathbf{C}_2) = \{\mathbf{0}\}$.
(4) Let $\mathcal{L}_i = \mathrm{Im}(\mathbf{C}_i)$, $i = 1, 2$ and show that

$$\underset{\substack{\shortparallel \\ \mathrm{Hom}(\mathcal{V}, \mathbb{R}^2)}}{\mathcal{L}(\mathcal{V}, \mathbb{R}^2)} = \mathcal{L}_1 + \mathcal{L}_2.$$

17. Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. A **complex structure** on $\mathcal{V}$ is a linear transformation

$$\mathbf{J} : \mathcal{V} \longrightarrow \mathcal{V}$$

such that for all $\mathbf{A}$ in $\mathcal{V}$,

$$\mathbf{J}^2(\mathbf{A}) = -\mathbf{A}.$$

(1) Show that

$$\mathbf{J}(x, y) = (-y, x)$$

is a complex structure on $\mathbb{R}^2$.

(2) Show that

$$\mathbf{J}(x, y) = (y, -x)$$

is a different complex structure on $\mathbb{R}^2$.

PROOF: Suppose that $ad - bc \neq 0$. Let

$$\mathbf{B} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Then

$$\begin{aligned}\mathbf{BA} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{bmatrix} \\ &= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}.\end{aligned}$$

A completely analogous computation shows that $\mathbf{AB} = \mathbf{I}$, and therefore $\mathbf{A}$ is nonsingular with

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Suppose conversely that $\mathbf{A}$ is nonsingular, but that $ad - bc = 0$. We will deduce a contradiction. Let

$$\mathbf{C} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Then computing as above,

$$\mathbf{CA} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\mathbf{I} = \mathbf{0}.$$

This gives the equation

$$\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AA}^{-1}) = (\mathbf{CA})\mathbf{A}^{-1} = \mathbf{0A}^{-1} = \mathbf{0}.$$

Therefore,

$$\mathbf{C} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so that

$$a = 0, \qquad b = 0, \qquad c = 0, \qquad d = 0.$$

But then $\mathbf{A} = \mathbf{0}$ also, so

$$\mathbf{I} = \mathbf{AA}^{-1} = \mathbf{0A}^{-1} = \mathbf{0},$$

so

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and hence $1 = 0$, which is impossible. □

One might wonder how the above result was originally discovered. Did somebody just make a lucky guess? Perhaps, but a more logical development may be found in Chapter 13.

10.5 Exercises

1. For each of the following linear transformations of $\mathbb{R}^3$ to $\mathbb{R}^3$ calculate the matrix relative to the standard basis:

(1) $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{T}(x, y, z) = (x + y + z, 0, 0)$.
(2) $\mathbf{Q} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{Q}(x, y, z) = (x, x + y, x + y + z)$.
(3) $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{F}(x, y, z) = (x + 2y + 3z, 2x + y, z - x)$.
(4) $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{G}(x, y, z) = (y - z, x + y, z - 2x)$.

2. Calculate the matrices of the linear transformations

$$\mathbf{T} \cdot \mathbf{Q} : \mathbb{R}^3 \to \mathbb{R}^3$$
$$\mathbf{F} \cdot \mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$$
$$\mathbf{Q} \cdot \mathbf{F} \cdot \mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$$

where $\mathbf{T}$, $\mathbf{Q}$, $\mathbf{F}$, and $\mathbf{G}$ are as in Exercise 1.

3. Suppose that the matrix of the linear transformation $\mathbf{S} : \mathbb{R}^3 \to \mathbb{R}^3$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

relative to the standard basis of $\mathbb{R}^3$. Calculate $\mathbf{S}(1, 2, 3)$.

4. Let $\mathbf{P} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$\mathbf{P}(x, y, z) = (x, y, 0).$$

Calculate the matrix for $\mathbf{P}$ and for $\mathbf{P}^2$ relative to the standard basis of $\mathbb{R}^3$.

5. Let $\mathbf{S}, \mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations with matrices $\mathbf{A}$ and $\mathbf{B}$ relative to the standard basis of $\mathbb{R}^3$. What is the matrix of $\mathbf{S} + \mathbf{T}$ relative to this basis?

6. Let $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear extension of

$$\mathbf{T}(\mathbf{E}_1) = (-1, 0, 2),$$
$$\mathbf{T}(\mathbf{E}_2) = (1, 1, -1),$$
$$\mathbf{T}(\mathbf{E}_3) = (1, -3, 4).$$

Calculate the matrix of $\mathbf{T}$ relative to the standard basis.

7. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear extension of

$$T(1, 1) = (1, -1),$$
$$T(1, -2) = (2, 2).$$

Calculate the matrix of the linear transformation T relative to the standard basis.

8. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z) = (0, y, z).$$

Calculate the matrix of T relative to the standard basis of $\mathbb{R}^3$.

9. Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the linear transformation defined by

$$T(x, y, z, w) = (0, x, y, z).$$

Calculate the matrix of T relative to the standard basis. Calculate the matrices of T^2, T^3, T^4 relative to the standard basis.

10. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation whose matrix relative to the standard basis is

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

(1) Calculate $T(\mathbf{E}_1)$, $T(\mathbf{E}_2)$, $T(\mathbf{E}_3)$.
(2) Let r be a real number. Calculate the matrix of rT relative to the standard basis.

11. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation whose matrix relative to the standard basis is

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

(1) Calculate $T(\mathbf{E}_1)$ and $T(\mathbf{E}_2)$.
(2) Find $\mathbf{A}_1, \mathbf{A}_2$ satisfying $T(\mathbf{A}_i) = \mathbf{E}_i$, $i = 1, 2$.

12. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation whose matrix relative to the standard basis is

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

(1) Calculate $T(\mathbf{E}_1)$, $T(\mathbf{E}_2)$, $T(\mathbf{E}_3)$, and $T(1, 2, 3)$.
(2) Is T surjective? (Check whether $(1, 0, 0)$ belongs to $\mathrm{Im}(T)$.)

13. Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathbb{R}^3$ and write

$$\mathbf{A}_j = (a_{1,j}, a_{2,j}, a_{3,j}), \qquad j = 1, 2, 3.$$

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear extension of the function $T(\mathbf{E}_j) = \mathbf{A}_j$, $j = 1, 2, 3$. Show that the matrix of T relative to the standard basis of $\mathbb{R}^3$ is $(a_{i,j})$.

14. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation and $\mathbf{A} = (a_{i,j})$ its matrix with respect to the standard basis of $\mathbb{R}^3$. Regard the columns of $\mathbf{A}$ as vectors in $\mathbb{R}^3$. Call these three vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$. Show that $\mathrm{Im}(T) = \mathcal{L}(\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\})$.

15. If $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the linear transformation with matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

with respect to the standard basis of $\mathbb{R}^3$, show that T is an isomorphism.

16. Perform the following matrix computations:

(a) $2\begin{bmatrix} 3 & 1 & 4 & -1 \\ 2 & 0 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & -1 & 7 \\ -1 & -2 & 0 & -4 \end{bmatrix}$.

(b) $2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 8\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

17. Perform the following matrix multiplications:

(a) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.