**A**: Linear Algebra and Its Applications by Gilbert Strang, 4th Edition

**B**: Linear Algebra by Larry Smith, 3rd Edition

**C**: Introduction to Linear Algebra by Gilbert Strang, 3rd Edition

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numbers, which we denote by \( \mathbb{C} \). A vector space with complex scalars is called a complex vector space. The axioms for a complex vector space are exactly the same as for a real vector space except that the numbers (= scalars) are to be complex. The generic example of a complex vector space is the complex Cartesian space \( \mathbb{C}^k \) of \( k \)-tuples \( A = (a_1, \ldots, a_k) \) of complex numbers, where for vectors \( A = (a_1, \ldots, a_k) \) and \( B = (b_1, \ldots, b_k) \) and for scalars \( r \in \mathbb{C} \), vector addition and scalar multiplication are given by

\[
A + B = (a_1 + b_1, \ldots, a_k + b_k)
\]

and

\[
rA = (ra_1, \ldots, ra_k).
\]

In the next few chapters we will study only real vector spaces, indicating where necessary the modifications required in the complex case.

### 2.4 Exercises

1. Let \( A = \{(2a, a) \mid a \in \mathbb{R}\}, \) \( B = \{(b, b) \mid b \in \mathbb{R}\}. \) Find \( A \times B \) and \( A \cap B. \)

2. Denote by \( \mathbb{Z} \) the set of all integers and let \( A = \{(2n, n) \mid n \in \mathbb{Z}\}, B = \{(k + 1, k) \mid k \in \mathbb{Z}\}. \) Find \( A \cap B, A \cup B. \)

3. If \( A \subseteq B, B \subseteq C, \) then \( A \subseteq C. \)

4. If \( A \subseteq B, A \subseteq C, \) then \( A \subseteq B \cup C. \)

5. If \( A \supseteq B, A \supseteq C, \) then \( A \supseteq B \cup C. \)

6. Show that \( A \cap (B \cap C) = (A \cap B) \cap (A \cap C) \) and \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \) where \( A, B, C \) are sets.

7. Let \( A = \{x \in \mathbb{R} \mid |x| > 1\}, B = \{x \in \mathbb{R} \mid -2 < x < 3\}. \) Find \( A \times B \) and \( A \cap B. \)

8. Let \( \mathbb{V}' \) be a vector space. Prove each of the following statements:
   (a) If \( A = \mathbb{V}' \) and \( a \) is a number, then \( aA = 0 \) if and only if \( a = 0 \) or \( A = 0, \) or both.
   (b) If \( A \subseteq \mathbb{V}' \) and \( a \) is a number, then \( aA = A \) if and only if \( a = 1 \) or \( A = 0, \) or both.

9. Let \( \mathbb{V}' \) be the set of all ordered pairs of real numbers \( (a, b). \) Define an addition for the elements \( \mathbb{V}' \) by the rule
   \[
   (a, b) \oplus (c, d) = (a + c, b + d),
   \]

   and a multiplication of elements of \( \mathbb{V}' \) by numbers by the rule
   \[
r \cdot (c, d) = (rc, rd).
   \]

Is \( \mathbb{V}' \) with these two operations a vector space? Justify your answer.

10. Let \( \mathbb{V}' \) and \( \mathbb{V}'' \) be vector spaces. Denote by \( \mathbb{V}' \oplus \mathbb{V}'' \) the set of all ordered pairs \( (A, B) \), where \( A \in \mathbb{V}' \) and \( B \in \mathbb{V}'' \). Define an addition for the elements of \( \mathbb{V}' \oplus \mathbb{V}'' \) by the rule
   \[
   (A, B) + (C, D) = (A + C, B + D),
   \]

   and a multiplication of elements of \( \mathbb{V}' \) by numbers by the rule
   \[
r \cdot (A, B) = (rA, rB).
   \]

Is \( \mathbb{V}' \oplus \mathbb{V}'' \) with these two operations a vector space? Justify your answer. (Beware of the fact that + and \( \cdot \) are used with multiple meanings in the preceding equations. \( \mathbb{V}' \oplus \mathbb{V}'' \) is called the direct sum of \( \mathbb{V}' \) and \( \mathbb{V}'' \).)

11. A translation of the plane \( \Pi \) is a function \( T : \Pi \rightarrow \Pi \) with the following two properties:
   (i) There is a constant \( k \), called the length of the translation, such that for every point \( p \in \Pi \) the distance from \( p \) to \( T(p) \) is equal to \( k \).
   (ii) For any two points \( p, q \in \Pi \), the distance from \( p \) to \( q \) is the same as the distance from \( T(p) \) to \( T(q) \).

   Introduce Cartesian coordinates in \( \Pi \) and show:
   (a) If \( T \) is a translation, then for every \( p = (x, y) \in \Pi, T(p) = (x + l_1, y + l_2), \) where \( T(0, 0) = (l_1, l_2). \) \( T \) is said to be the translation by \( l = (l_1, l_2). \)
   (b) If \( l = (l_1, l_2) \in \Pi, \) show that
      \[
      T(p) = (x + l_1, y + l_2), \quad p = (x, y) \in \Pi
      \]
      defines a translation of \( \Pi. \)

   If \( T, S \) are translations, define their sum \( T \oplus S \) by
   \[
   (T \oplus S)(p) = T(S(p)).
   \]
   (c) Show that \( T \oplus S \) is again a translation.

   If \( T \) is the translation by \( l = (l_1, l_2) \) and \( a \) is a number, define \( a \cdot T \) to be the translation by \( (al_1, al_2). \)
   (d) Show that the set of translations with the addition \( \oplus \) and scalar multiplication \( \cdot \) is a vector space.

12. Show that if a vector space contains two elements, then it contains infinitely many.

13. Let \( A, B \) be elements of the vector space \( \mathbb{V}'. \) Show that there exists a unique element \( X \) of \( \mathbb{V}' \) such that \( A + X = B. \)
4. Subspaces

**Example 3:** Let $S$ be the set consisting of the three elements $a, b, c$ and consider all the functions $f$ in $\text{Fun}(S)$ with the property that

$$f(a) + f(b) + f(c) = 0.$$ 

Call this set of functions $\mathcal{N}$. If $f, g \in \mathcal{N}$ and $r \in \mathbb{R}$, then by the definition of vector addition and scalar multiplication in $\text{Fun}(S)$ we find that

$$(f + g)(a) + (f + g)(b) + (f + g)(c)$$

$$= f(a) + g(a) + f(b) + g(b) + f(c) + g(c)$$

$$= f(a) + f(b) + f(c) + g(a) + g(b) + g(c) = 0,$$

$$(rf)(a) + (rf)(b) + (rf)(c) = r \cdot f(a) + r \cdot f(b) + r \cdot f(c)$$

$$= r \cdot (f(a) + f(b) + f(c)) = r \cdot 0 = 0,$$

and therefore $\mathcal{N}$ is a linear subspace of $\text{Fun}(S)$.

You might want to think through what happens in Examples 2 and 3 if the condition that something be zero is replaced by the same thing being, say, $\pi$.

### 4.3 Exercises

1. Which of the following collections of vectors in $\mathbb{R}^3$ are linear subspaces?

   (1) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0\}$

   (2) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 0\}$

   (3) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 = 0\}$

   (4) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 = 1\}$

   (5) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 \geq 0\}$

2. Determine the subspace of $\mathbb{R}^3$ that is the linear span of the three vectors $(1, 0, 1), (0, 1, 0), (0, 1, 1)$.

3. Repeat Exercise 2 for $(1, 0, 0), (0, 1, 0), (1, 1, 1)$.

4. Let $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = a\}$ for $a \in \mathbb{R}$ fixed. Show that $\mathcal{U}$ is a vector subspace if and only if $a = 0$.

5. Show that $\mathcal{P}_s(\mathbb{R})$ is a linear subspace of $\mathcal{P}_r(\mathbb{R})$ whenever $s \leq r$.

6. Show that $\mathcal{P}_1(\mathbb{R})$ is always a subspace of $\mathcal{P}(\mathbb{R})$.

7. What is the span of $\{1 + x, 1 - x\}$ in $\mathcal{P}(\mathbb{R})$?

8. What is the span of $\{1, x^2, x^4\}$ in $\mathcal{P}(\mathbb{R})$?

9. Find a vector that spans the subspace $2x - 3y = 0$ of $\mathbb{R}^2$.

10. Find a pair of vectors that span the subspace $x + y - 2z = 0$ of $\mathbb{R}^3$.

11. Suppose $S, T$ are subspaces of $\mathcal{V}$ and $S \cap T = \emptyset$. Show that every vector in $S + T$ can be written uniquely in the form $A + B$, with $A \in S$, $B \in T$. Construct an example to show that this is false if $S \cap T \neq \emptyset$.

12. Show that any nonzero vector spans $\mathbb{R}^1$.

13. Show that the two sets of vectors

   $$\{A = (1, 1, 0), B = (0, 0, 1)\}$$

   and

   $$\{C = (1, 1, 1), D = (-1, -1, 1)\}$$

span the same subspace of $\mathbb{R}^3$.

14. Let $\mathcal{V}$ be the set of pairs of numbers $A = (a_1, a_2)$. If $A, B \in \mathcal{V}$ define

   $$A + B = (a_1 + b_1, a_2 + b_2),$$

   where $B = (b_1, b_2)$. If $a$ is a number define

   $$aA = (aa_1, 0).$$

Is $\mathcal{V}$ a vector space? Why?

15. Let $\ell_\infty$ be the vector space of bounded sequences (see Chapter 3, Exercise 16). Which of the following collections of vectors in $\ell_\infty$ are linear subspaces?

   (1) $\{(a_0, a_1, \ldots, a_n) \mid |a_i| < 5 \forall i \in \mathbb{N}_0\}$

   (2) $\{(a_0, a_1, \ldots, a_n) \mid a_i = 0 \forall i > i_0, i_0 \in \mathbb{N}_0\}$

   (3) $\{(a_0, a_1, \ldots, a_n) \mid a_i = a_j \forall i \neq j\}$

   (4) $\{(a_0, a_1, \ldots, a_n) \mid a_i \in \mathbb{Z} \forall i \in \mathbb{N}_0\}$

16. Suppose $\mathcal{V}$ is a vector space and $E, F$ are subspaces of $\mathcal{V}$. Show that $E \subseteq F \subseteq L(E) \subseteq L(F)$.

17. Let $\mathcal{V}$ be a vector space and $E, F \subseteq \mathcal{V}$. Suppose $L(E) \subseteq L(F)$. Is it true that $E \subseteq F$?

18. Suppose that $\mathcal{V}$ is a vector space and $E \subseteq \mathcal{V}$. If $\mathcal{U}$ is a subspace containing $E$, show that $\mathcal{U}$ contains $L(E)$.

The vectors in $\mathcal{P}(\mathbb{R})$ are polynomials $p(x) = a_0 + a_1x + \ldots + a_nx^n$ of degree less than or equal to $n$, that is, $\deg p \leq n$.

The scalar multiplication is the ordinary addition of polynomials. The scalar multiplication is the ordinary product of a polynomial by a number.

$\mathcal{P}(\mathbb{R})$ is defined similarly, but we remove the restriction on the degree of the polynomials.
19. Suppose \( \mathcal{V} \) is a vector space and \( E \subseteq \mathcal{V} \). Show that \( L(E) = \{ \mathcal{U} \mid \mathcal{U} \text{ is a subspace of } \mathcal{V} \text{ and } \mathcal{U} \text{ contains } E \} \).

20. For any subspace \( \mathcal{U} \) of \( \mathcal{V} \) show that \( \mathcal{U} + \mathcal{U} = \mathcal{U} \).

21. Let \( S \) and \( T \) be linear subspaces of the vector space \( \mathcal{V} \). Show that \( S \cap T = \mathcal{V} \) if and only if \( S \) or \( T \), or both, is equal to \( \mathcal{V} \).

22. Find all the linear subspaces of \( \mathbb{R}^2 \).

23. Find all the linear subspaces of \( \mathbb{R}^3 \).

24. Show that a subset \( E \) of a vector space \( \mathcal{V} \) that does not contain \( 0 \) is not a subspace of \( \mathcal{V} \).

25. Let \( E \) be the subset of \( \mathbb{R}^2 \) defined by \( E = \{(x, y) \mid x \geq 0, y \in \mathbb{R} \} \). Is \( E \) a subspace of \( \mathbb{R}^2 \)?

26. Let \( E = \{(x, 2x + 1) \mid x \in \mathbb{R} \} \), so \( E \) is a subset of \( \mathbb{R}^2 \). Is \( E \) a subspace of \( \mathbb{R}^2 \)?

27. (1) Let \( E = \{(2a, a) \mid a \in \mathbb{R} \} \). Is \( E \) a subspace of \( \mathbb{R}^2 \)?

(2) Let \( B = \{(b, b) \mid b \in \mathbb{R} \} \). Is \( B \) a subspace of \( \mathbb{R}^2 \)?

(3) What is \( E \cap B \)?

(4) Is \( E \cap B \) a subspace of \( \mathbb{R}^2 \)?

(5) What is \( E \cap B \)?

28. Let \( S \) and \( T \) be subspaces of \( \mathcal{V} \). Prove:

(1) \( S + T = L(S \cap T) \).

(2) \( S \cap (S + T) = S \).

(3) \( S + T = T + S \).

(4) If \( S \subseteq T \), then \( S + T = T \).

29. Let \( \mathcal{V} \) be a vector space, \( A, B, C \in \mathcal{V} \). Suppose \( A + B + C = 0 \). Show that \( L(A, B) = L(B, C) \).

30. Let \( \mathcal{V} \) be a vector space and \( \mathcal{W} \) a subspace of \( \mathcal{V} \). Show that:

\[
\{ A \in \mathcal{W} \mid A \in L(A, B) \}
\]

is not a vector subspace of \( \mathcal{V} \).

31. Let \( \mathcal{V} \) be a vector space, \( \mathcal{W} \) a subspace of \( \mathcal{V} \), and \( A, B \in \mathcal{V} \). Assume that

\[
A \in \mathcal{W} \text{ but } A \notin L(\mathcal{W} \cup \{B\}).
\]

Show that \( B \in L(\mathcal{W} \cup \{A\}) \).

32. Let \( S, T, \mathcal{U} \) be subspaces of a vector space \( \mathcal{V} \). Is it always true that \( S \cap (T + \mathcal{U}) = (S \cap T) + (S \cap \mathcal{U}) \)?

33. Can it happen for two subspaces \( S \) and \( T \) of \( \mathcal{V} \) that \( S \cap T = \emptyset \)?

34. Can you find two vectors \( A, B \in \mathbb{R}^2 \) such that \( L(A, B) = \mathbb{R}^3 \)?

35. Let \( \mathcal{V} \) be a vector space, \( \mathcal{W} \) a subspace of \( \mathcal{V} \), \( S \) a set, and \( s \in S \). Show that

\[
\mathcal{U} = \left\{ f \in L(S, \mathcal{V}) \left| f(s) \in \mathcal{W} \right. \right\}
\]

is a subspace of \( L(S, \mathcal{V}) \).

36. Let \( S \) be a set and \( T \subseteq S \). Is it true that

\[
\text{Fun}(S) = \text{Fun}(S, T) + \text{Fun}(S, S - T) \]

37. Let \( \mathcal{V} \) be a vector space and \( S \) a set. Let \( \text{Fun}(S, \mathcal{V}) \) be defined as in Chapter 3 Exercise 10:

(1) If \( \mathcal{W} \) is a subspace of \( \mathcal{V} \), show that \( \text{Fun}(S, \mathcal{W}) \) is a subspace of \( \text{Fun}(S, \mathcal{V}) \).

(2) If \( S \) and \( T \) are subspaces of \( \mathcal{V} \), show that

\[
\text{Fun}(S, S) + \text{Fun}(S, T) = \text{Fun}(S, S + T).
\]
Example 1: Let \( E = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) \} \) be vectors in \( \mathbb{R}^3 \). Find a linearly independent set \( F \) that is a subset of \( E \) such that \( \ell(F) = \ell(E) \).

Solution: The proof of Theorem 5.2.4 suggests that we look for a vector in \( E \) linearly dependent on the remaining vectors of \( E \) and throw it away. If that doesn't lead to a linearly independent set with the same span, do it again, etc. Observe that
\[
(1, 1, 1) = 1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1),
\]
so that setting \( F = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \} \) we have \( \ell(F) = \ell(E) \). The set \( F \), however, is a linearly independent set of vectors and so we are done.

Alternative Solution: Proceeding as in the previous solution, we note that \( (1, 0, 0) \) is linearly dependent on \( (1, 1, 1) \) and \( (0, 1, 0) \), and setting \( H = \{ (1, 1, 1), (3, 1, 0), (0, 0, 1) \} \) we have \( \ell(H) = \ell(E) \). The set \( H \), however, is a linearly independent set of vectors and so we are done.

Moral: If \( E \) is a finite set of vectors in \( \mathbb{R}^3 \), then there need not exist a unique subset \( F \subseteq E \) with \( F \) linearly independent and \( \ell(F) = \ell(E) \).

Example 2: Let \( S \) be a finite set and \( X = \{ \chi_s \mid s \in S \} \subseteq \text{Fun}(S) \). These characteristic functions span \( \text{Fun}(S) \) because for any \( f \in \text{Fun}(S) \) we have the equality
\[
f = \sum_{s \in S} f(s) \chi_s.
\]
This is seen by evaluating the right-hand side at an arbitrary element \( t \in S \), giving
\[
\left[ \sum_{s \in S} f(s) \chi_s \right] (t) = \sum_{s \in S} f(s) \cdot \chi_s(t) = \sum_{s \in S} f(s) \cdot 0 + f(t) = f(t)
\]
as required.

Remark: Note that none of the functions \( \chi_s \) for \( s \in S \) can be left out and still leave us with a set of functions that span \( \text{Fun}(S) \), for the only way to write \( \chi_s \) as a linear combination of characteristic functions is as
\[
\chi_s = \sum_{t \in S} \delta_{s,t} \chi_t = 1 \cdot \chi_t.
\]

Remark: Example 2 cannot be extended to infinite sets. For if \( S \) is an infinite set then the function
\[
f : S \to \mathbb{R}
\]
defined by \( f(s) = 1, s \in S \) is not a (finite) linear combination of characteristic functions.

5.3 Exercises

1. Which of the following sets of vectors in \( \mathbb{R}^3 \) are linearly dependent and which are linearly independent?
   \[
   E = \{ (1, 1, 1), (0, 1, 0), (1, 0, 1) \}.
   \]
   \[
   F = \{ (1, 1, 1), (1, 1, 0), (1, 0, 0) \}.
   \]
   \[
   G = \{ (1, 1, 1), (1, 0, 0), (1, 0, 1) \}.
   \]
   \[
   H = \{ (1, 0, 0), (0, 1, 0), (1, 1, 1) \}.
   \]
   \[
   K = \{ (1, 1, 1), (0, 1, 0), (0, 0, 1) \}.
   \]

2. Which of the following sets of vectors in \( \mathbb{R}^3 \) are linearly dependent and which are linearly independent?
   \[
   E = \{ (1, x, x^2) \}.
   \]
   \[
   F = \{ (1 + x, 1 - x, x^2, 1) \}.
   \]
   \[
   G = \{ x^2 - 1, x + 1, x^2 - x, x^2 + x \}.
   \]
   \[
   H = \{ x - x^2, x + 1, x^2 - x \}.
   \]
   \[
   K = \{ 1, 1 - x, 1 - x^2 \}.
   \]

3. Show that the set of vectors in \( \mathbb{R}^3 \)
   \[
   E = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) \}
   \]
is linearly dependent but that any set of three of them is linearly independent.

4. Let \( \ell_\infty \) be the vector space of bounded sequences of numbers (see Chapter 3 Exercise 16). Which of the following collections of vectors in \( \ell_\infty \) are linearly independent?
   \[
   (1) \{ (1, 1, 1, 1, \ldots), (2, 1, 1, 1, \ldots), (1, 1, 1, 1, \ldots), (0, 1, 1, 1, \ldots) \}.
   \]
   \[
   (2) \{ (1, 1, 1, 1, \ldots), (0, 1, 1, 1, \ldots), (0, 1, 1, 1, \ldots), (0, 1, 1, 1, \ldots) \}.
   \]
   \[
   (3) \{ (1, 1, 1, 1, 0, 0, 0, 0, \ldots), (1, 1, 1, 1, 0, 0, \ldots), (1, 1, 1, 1, 0, 0, \ldots) \}.
   \]
   \[
   (4) \{ (1, 1, 1, 1, 0, 0, 0, \ldots), (1, 1, 1, 1, 0, 0, \ldots), (1, 1, 1, 1, 0, 0, \ldots) \}.
   \]
   \[
   (5) \{ (1, 1, 1, 1, 0, 0, \ldots), (0, 1, 1, 1, 0, 0, \ldots), (0, 1, 1, 1, 0, 0, \ldots), \ldots, (0, 1, 1, 1, 0, 0, \ldots) \}.
   \]
5. Linear Independence and Dependence

5. Let \( U \) be the subspace of \( \mathbb{R}^5 \) spanned by the vectors
\[
E = \{ (1, 1, 0, 0, 1), (1, 1, 0, 1, 1), (0, 1, 1, 0, 1), (2, 1, 1, 0, 0) \}.
\]
Find a linearly independent subset \( F \) of \( E \) with \( \ell(F) = U \).

6. Let \( U \) be the subspace of \( \mathbb{R}^5 \) spanned by
\[
F = \{ x^3, x^3 - x^2, x^3 + x^2, x^3 - 1 \}.
\]
find a linearly independent subset \( F \) of \( E \) spanning \( U \). (HINT: Try to use Theorem 5.2.4.)

7. For each of the linearly dependent sets \( S \) in Exercise 4 use Theorem 5.2.4 to find a linearly independent subset of vectors with the same span as \( S \).

8. Under what conditions on the numbers \( a \) and \( b \) are the vectors \( (1, a), (1, b) \) linearly independent in \( \mathbb{R}^2 \).

9. Suppose that \( E \) and \( F \) are sets of vectors in \( \mathcal{V} \) with \( E \cap F \neq \emptyset \). \( E \) is linearly dependent, then so is \( F \).

10. Suppose that \( E \) and \( F \) are sets of vectors in \( \mathcal{V} \) with \( E \cap F \neq \emptyset \). \( F \) is linearly independent then so is \( E \).

11. Show that functions \( e^x \) and \( e^{2x} \) form a set of linearly independent vectors in \( C([0, \infty)) \).

12. Show that
\[
\left\{ \cos x, \sin x, \sin \left( x + \frac{\pi}{2} \right) \right\}
\]
is a set of linearly dependent vectors in \( C([0, \infty)) \).

13. Is the pair of complex numbers \( \alpha + \beta i \), \( \alpha - \beta i \) a set of linearly independent vectors in \( \mathbb{C} \), where \( \alpha, \beta \) are any nonzero real numbers?

14. Show that the set of polynomials \( E = \{ x^3, 1 + x^2 \} \) is a set of linearly independent vectors in \( \mathcal{P}_2(\mathbb{R}) \). What is the space spanned by \( E \)? Is \( \ell(E) = \mathcal{P}_2(\mathbb{R}) \)? If not, find a vector in \( \mathcal{P}_2(\mathbb{R}) \) that does not belong to \( \ell(E) \), and show that together with \( E \) they form a set of linearly independent vectors in \( \mathcal{P}_2(\mathbb{R}) \).

15. Let \( F \) be a set of vectors in the vector space \( \mathcal{V} \). Show that \( F \) is linearly independent if and only if for every proper subset \( E' \) of \( F \), \( \ell(E') \neq \ell(E) \). Equivalently, show that \( F \) is linearly dependent if and only if there exists a proper subset \( E' \) of \( F \) such that \( \ell(E') = \ell(E) \).

16. Let \( S = \{ x_1, x_2, x_3 \} \) and consider the functions \( f, g, h \in \text{Fun}(S) \) defined by
\[
\begin{align*}
f(x_1) &= 0, \quad f(x_2) = 1, \quad f(x_3) = 1, \\
g(x_1) &= 1, \quad g(x_2) = 0, \quad g(x_3) = 1, \\
h(x_1) &= 1, \quad h(x_2) = 1, \quad h(x_3) = 0.
\end{align*}
\]
Does \( \{ f, g, h \} \) form a linearly independent set?

17. Let \( E', E'' \) be linearly independent sets of vectors in \( \mathcal{V} \). Show that:
\( (1) \ E' \cap E'' \) is linearly independent.
\( (2) \ E' \cup E'' \) is linearly independent if and only if \( \ell(E') \cap \ell(E'') = \{ \emptyset \} \).

18. Let \( A, B, C \in \mathcal{V} \) be linearly independent vectors. Show that the vectors \( A + B, B + C, C + A \) are linearly independent.

19. Let \( \mathcal{V} \) be a vector space and \( A, B, C, D \in \mathcal{V} \). Show that the vectors
\[
A + B + C, \quad 2A + 2B + C, \quad A + B + C, \quad A + C + D
\]
are linearly dependent. Find a maximal linearly independent subset of these four vectors.

20. Let \( S, T \) be subspaces of a vector space \( \mathcal{V} \), and \( S \neq S', T \neq T', S \cap T \neq \emptyset \). Assume \( S \cap T = \{ 0 \} \). Show that \( \{ S, T \} \) is a linearly independent set of vectors.

21. Show that any two vectors in \( \mathbb{R}^3 \) must be linearly dependent.

22. Let \( S, T \) be subspaces of \( \mathcal{V} \) and \( S \) a set. Suppose \( f \in \text{Fun}(S, S), g \in \text{Fun}(S, T), \) and \( S \cap T = \emptyset \). Show that regarded as vectors in \( \text{Fun}(S, T) \), the vectors \( f \) and \( g \) are linearly independent.

23. Let \( A, B, C \in \mathcal{V} \). Define \( E = \{ (A, B) \} \). Show that \( A \) and \( B \) are linearly independent if and only if \( A + C + B \) and \( C + B \) are linearly independent.

24. Let \( f_1, \ldots, f_n \in \mathcal{P}_n(\mathbb{R}) \) be polynomials of positive degree. Suppose \( f_1, \ldots, f_n \) are linearly independent. Does it follow that their derivatives \( df_1/dx, \ldots, df_n/dx \) are linearly independent?

25. Let \( S \) be a set. A permutation \( \alpha \) of \( S \) is a bijective (i.e., one to one and onto) mapping \( \alpha : S \to S \). For a permutation \( \alpha \) of \( S \) and \( f \in \text{Fun}(S) \)
\[
f : S \to \mathbb{R} \text{ by } f(\alpha(s)) \text{.}
\]
Suppose \( f_1, \ldots, f_n \in \text{Fun}(S) \) are linearly independent. Show for any permutation \( \alpha \) that \( (f_1), \ldots, (f_n) \) are also linearly independent.

26. If \( f_1, \ldots, f_n \in \mathcal{P}_n(\mathbb{R}) \) are linearly dependent then so are their derivatives \( df_1/dx, \ldots, df_n/dx \).
EXAMPLE 6: For what values of \( r \) are the vectors \((r, 1, 1), (1, r, 1), (1, 1, r)\) a basis for \( \mathbb{R}^3 \)?

**Solution:** Since \( \dim(\mathbb{R}^3) = 3 \), it will suffice to determine when the vectors in question are linearly independent. So suppose that

\[
a(r, 1, 1) + b(1, r, 1) + c(1, 1, r) = 0
\]

is a linear relation. Then

\[
(ar + b + c, a + br + c, a + b + cr) = 0 = (0, 0, 0),
\]

so taking coordinates gives

\[
\begin{align*}
ar + b + c &= 0, \\
a + br + c &= 0, \\
a + b + cr &= 0.
\end{align*}
\]

From the last equation we get

\[(*) \quad cr = -(a + b).\]

Multiplying the second equation by \( r \) and substituting gives

\[
0 = ar + b(r^2) - (a + b) = a(r - 1) + b(r^2 - 1)
\]

\[
= (r - 1)(a + b(r - 1)).
\]

So if \( r - 1 \neq 0 \), we may divide by \( r - 1 \) and solve for \( a \) to get

\[(***) \quad a = -(r + 1)b.\]

Multiplying the first equation by \( r \) and substituting in \((*)\) and \((***)\), we obtain

\[
0 = [-r(r + 1)b]r^2 + br - (a + b)
\]

\[
= -r^2(r + 1)b + b - [-1 + b + b]
\]

\[
= -r^2(r + 1) + r + (r + 1) - 1]
\]

\[
= -r^2 - 2r + r^2 + r + r = -(r^2 + r - 2)
\]

\[
= -(r^2 + r - 2)b = -r(r + 2)(r - 1)b.
\]

So assuming \( r \neq 0, -2, 1 \) we may divide by \( -r + 2(r - 1) \) and conclude that \( b = 0 \), whence \( a = 0 \) by \((***)\) and \( c = 0 \) by \((*)\). Therefore, the vectors are independent when \( r \neq 0, 1, -2 \). For \( r = 0 \), the original system simplifies to

\[
0 = b + c,
\]

\[
0 = a + c,
\]

\[
0 = a + b.
\]

Substituting the first equation into the second gives

\[
0 = a - b
\]

and adding to the third gives

\[
0 = 2a,
\]

giving \( a = 0 \) and \( b = 0, c = 0 \), so again the vectors are independent. For \( r = 1 \), the three vectors are the same vector three times, so they cannot be a basis for \( \mathbb{R}^3 \). And finally, for \( r = -2 \), there is the linear relation

\[
(-2, 1, 1) + (1, -2, 1) + (1, 1, -2) = (0, 0, 0),
\]

so the vectors are not linearly dependent. In conclusion, the vectors \((r, 1, 1), (1, r, 1), (1, 1, r)\) are a basis for \( \mathbb{R}^3 \) if and only if \( r \neq 1, -2 \).

**6.4 Exercises**

1. Which of the following sets of vectors are bases for \( \mathbb{R}^3 \)?

   \[ K = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}. \]

   \[ F = \{(1, 0, 1), (0, 1, 0), (1, 1, 1)\}. \]

   \[ G = \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}. \]

   \[ H = \{(1, -1, 0), (1, 0, -1), (-1, 0, 1)\}. \]

2. Which of the following sets of vectors are bases for \( \mathbb{P}_3(\mathbb{R}) \)?

   \[ E = \{1, 1 + x, 1 + x + x^2\}. \]

   \[ F = \{1, (x - 1), (x - 1)x - 2\}. \]

   \[ G = \{x^2 + x + 1, x^2 - x + 1, x^2 - x - 1\}. \]

   \[ H = \{x^2 + x + x^2 + 1, x + 1\}. \]

3. Find a basis for each of the following subspaces of \( \mathbb{R}^3 \):

   \[ U = \{(x, y, z) \mid x - z = 0\}. \]

   \[ S = \{(x, y, z) \mid x + y + z = 0\}. \]

   \[ T = \{(x, y, z) \mid x - y + z = 0\}. \]

   \[ W = \{(x, y, z) \mid x = 0 \text{ and } y + z = 0\}. \]
4. Find a basis for each of the following subspaces of \( \mathbb{P}_5(\mathbb{R}) \):
   \[ T = \left\{ p(x) \mid p(0) = 0 \right\} \]
   \[ S = \left\{ p(x) \left| \frac{d}{dx} p(x) = 0 \right\} \right. \]
   \[ T = \left\{ p(x) \mid p(x) = a_0 + a_1 x + a_3 x^3 \right\} \]
   \[ W = \left\{ p(x) \mid p(-x) = p(x) \right\} \]

5. Calculate the dimension of each of the subspaces in the preceding two exercises.

6. Let \( \ell_\infty \) denote the vector space of bounded sequences (see Chapter 3 Exercise 16). Which of the following sets of vectors are bases for \( \ell_\infty \)?
   \[ E = \{(1, 0, \ldots, 0, \ldots), (0, 1, 0, \ldots, 0, \ldots), \ldots, (0, 0, \ldots, 1, 0, \ldots) \} \]
   \[ F = \{(1, 1, \ldots, 1, \ldots), (0, 1, 1, 0, \ldots, 1, 1, \ldots), \ldots, (0, 0, \ldots, 0, 0, \ldots, 1, 1, 1) \} \]
   \[ G = \{(1, 0, \ldots, 0, \ldots), (0, 1/2, 0, \ldots, 0, \ldots), \ldots, (0, \ldots, 1/n, 0, \ldots) \} \]

7. For each of the following subspaces of \( \ell_\infty \), show that the subset is a linear subspace and find a basis.
   \[ A = \left\{ (a_0, a_1, \ldots) \mid a_i = a_j \forall i, j \in \mathbb{N} \right\} \]
   \[ S_m = \left\{ (a_0, a_1, \ldots, a_m) \mid a_i = 0 \forall i > n \right\} \text{ where } n \in \mathbb{N} \text{ is a fixed nonnegative integer.} \]

8. Suppose that \( \mathcal{V} \) is a finite-dimensional vector space and \( S \) is a linear subspace of \( \mathcal{V} \). Show that there exists a linear subspace \( T \) of \( \mathcal{V} \) such that \( S \cap T = \{0\} \) and \( S + T = \mathcal{V} \). (Hint: Study the proof of the basis extension theorem.)

9. Suppose that \( S \) and \( T \) are subspaces of a finite-dimensional vector space \( \mathcal{V} \), and \( S \cap T = \{0\} \). Show that
   \[ \dim(S + T) = \dim(S) + \dim(T) \]

10. Suppose that \( S \) and \( T \) are subspaces of a finite-dimensional vector space \( \mathcal{V} \). Show that
    \[ \dim(S + T) = \dim(S) + \dim(T) - \dim(S \cap T) \]
    (Hint: Choose a basis for \( S \cap T \), extend it to a basis for \( S \), and extend it to a basis for \( T \). Show that the result spans \( \mathcal{V} \) and count the number of vectors that occur twice.)

11. Suppose that \( S_1, \ldots, S_k \) are subspaces of a finite-dimensional vector space \( \mathcal{V} \). Find a formula for the dimension of \( S_1 + \cdots + S_k \). (Hint: Try the case \( k = 2 \).)

12. What is the dimension of \( \mathcal{C} \) as a vector space over \( \mathbb{R} \)? What is the dimension of \( \mathcal{C} \) as a vector space over \( \mathcal{C} \)?

13. Let \( S = \{(a, 0, 0)\} \) and \( T = \{(0, b, b)\} \), where \( a \) and \( b \) are real numbers. Show \( S \) and \( T \) are subspaces of \( \mathcal{R}^3 \). Find a basis for \( S + T \).

14. Let \( S \) be the subspace of \( \mathbb{R}^3 \) given by
   \[ S = \{(x, y, z) \mid y = z = 0\} \]
   Find a subspace \( T \) of \( \mathbb{R}^3 \) such that \( S \cap T = \{0\} \) and \( S + T = \mathbb{R}^3 \).

15. Under what conditions on the number \( a \) will the vectors
   \[ (a, 1, 0), (1, a, 1), (0, 1, a) \]
   be a basis for \( \mathbb{R}^3 \)?

16. Let \( A_1, \ldots, A_n \) be a basis for \( \mathbb{R}^n \). Set
    \[ S = \left\{ \sum_{i=1}^{n} a_i A_i \mid a_1, \ldots, a_n \in \mathbb{R} \text{ and all } a_i \neq 0 \right\} \]
    \[ T = \left\{ \sum_{i=1}^{n} a_i A_i \mid a_1, \ldots, a_n \in \mathbb{R} \text{ and } \sum_{i=1}^{n} a_i = 0 \right\} \]
    Show that \( S \) and \( T \) are vector subspaces and find bases for \( S, T, S \cap T \), and \( S + T \).

17. Let \( A_1, \ldots, A_n \) be vectors in \( \mathcal{V} \). Suppose that \( n = \dim(\mathcal{V}) \). Show that \( A_1, \ldots, A_n \) are linearly independent if and only if \( \langle A_1, \ldots, A_n \rangle \) has dimension \( n \).

18. The equation \( y = 3x \) defines a straight line in the plane. Show that if \( A, B \) are points on this line then \( A, B \) are linearly dependent vectors.

19. Let \( A, B, C, D \) be four distinct points on a plane \( \Pi \). Show that the vectors \( AB, AC, AD \) form a set of linearly independent vectors.

20. Let \( A, B, C \) be three points in a space. Show that \( A, B, C \) are not collinear (i.e., they all lie on the same line) if and only if \( \{AB, AC\} \) is a set of linearly independent vectors.
7.2 Exercises

1. Let \( A = (1, 0, 1), B = (-1, 1, 0), C = (0, 1, 1) \) be three vectors of \( \mathbb{R}^2 \). Show that \( \langle A, B \rangle = \langle B, C \rangle \).

2. \( \{ B_1, B_2 \} \) is a basis for \( \langle B_1, B_2 \rangle \) in Example 5 of this chapter. Find the coordinates (in components) of \( A_1 \) relative to this basis \( \{ B_1, B_2 \} \). Do the same problem for \( A_2 \) and \( A_3 \).

3. Let \( P = (a, b) \) be a vector in \( \mathbb{R}^2 \).
   (1) Show that \( a, b \) are the coordinates of \( P \) relative to the basis \( E_1 = (1, 0), E_2 = (0, 1) \).
   (2) Find the coordinates of \( P = (a, b) \) relative to the basis \( E_1 = (1, 0) \) and \( E_2 = (0, 2) \).
   (3) Find the coordinates of \( P \) relative to the basis \( G = (1, 1) \) and \( H = (-1, 2) \).

4. Are the polynomials \( \{ x + x^2, 1 + x^2 \} \) a set of linearly independent vectors of \( \mathbb{P}_3(\mathbb{R}) \)? If so, is \( \{ x + x^2, 1 + x^2 \} \) a basis for \( \mathbb{P}_3(\mathbb{R}) \)? What is the dimension of \( \mathbb{P}_3(x + x^2, 1 + x^2) \)?

5. The solution space \( S \) of \( x - 2y + 3z = 0 \) is a subspace of \( \mathbb{R}^3 \). Show that \( \dim(S) = 2 \). Find a basis for \( S \).

6. Let \( S \) be a set. A map \( \tau : S \rightarrow S \) is called an involution if \( \tau(\tau(s)) = s, \forall s \in S \). Let \( \tau \) be an involution on the set \( S \) and define \( \text{Fun}(S) = \{ f \in \text{Fun}(S) \mid f(\tau(s)) = f(s), \forall s \in S \} \). 
   (1) Show that \( \text{Fun}(S) \) and \( \text{Fun}(S) \) are subspaces of \( \text{Fun}(S) \).
   (2) Show that \( \text{Fun}(S) \cap \text{Fun}(S) = \{ 0 \} \).
   (3) Let \( f \in \text{Fun}(S) \) and define \( f_+ : S \rightarrow \mathbb{R} \) by \( f_+(s) = f(s) + f(\tau(s)) \).
   (4) Show that \( \text{Fun}(S) = \text{Fun}(S) + \text{Fun}(S) \).
   (5) If \( S \) is finite show that \( \dim(\text{Fun}(S)) + \dim(\text{Fun}(S)) = |S| \).

7. Let \( \mathcal{V} \) be a vector space, and \( S, T \) finite-dimensional subspaces of \( \mathcal{V} \). If \( S \) and \( T \) have the same dimension and \( S + T \), show that \( S = T \).

8. Let \( S = \{ s, t, u \} \) and let \( \tau : S \rightarrow S \) be the involution obtained by interchanging \( s \) and \( u \) and leaving \( t \) alone, that is \( \tau(s) = u, \tau(t) = t, \tau(u) = s \).

9. Find bases for \( \text{Fun}(S) \) and \( \text{Fun}(S) \) (see Exercise 6).

10. Let \( S, T, U \) be vector subspaces of the vector space \( \mathcal{V} \). Form the nine subspaces \( S, \ S \cap T, \ S \cap U, \ S + T, \ S \cap (T + U), \ S + (T \cap U), \ T + U, \ (S \cap T) + (S \cap U), \ (S + T) \cap (S + U) \).
    Show that for \( X, Y \) any two of these nine subspaces, either \( X \subseteq Y \) or \( Y \subseteq X \). Find an example with the maximum number of distinct subspaces among the nine.
9. Linear Transformations: Examples and Applications

**Proof**: The polynomial

\[ P_f(t) = \sum_{i=0}^{n} f(t_i) \Phi_i(t) \]

has degree \( n \) and agrees with \( f(t) \) at the points \( t_0, t_1, \ldots, t_n \). If \( Q_f(t) \) is any other polynomial of degree \( n \) with this property, then, since \( \Phi_0, \Phi_1, \ldots, \Phi_n \) is a basis for \( P_n(\mathbb{R}) \) (Proposition 9.2.1), it follows that

\[ Q_f = \sum_{j=0}^{n} a_j \Phi_j \]

for certain uniquely defined numbers \( a_0, a_1, \ldots, a_n \). Evaluating this equation at \( t_i \) yields

\[ f(t_i) = Q_f(t_i) = \sum_{j=0}^{n} a_j \Phi_j(t_i) = a_i, \]

and hence \( Q_f = P_f \), as required. \( \square \)

9.3 Exercises

1. Let \( k(x) \) be a fixed polynomial. Define functions

\[ M_{k(x)} : P(\mathbb{R}) \rightarrow P(\mathbb{R}) \]
\[ L_{k(x)} : P(\mathbb{R}) \rightarrow P(\mathbb{R}) \]

by

\[ M_{k(x)}(p(x)) = k(x)p(x) \]
\[ L_{k(x)}(p(x)) = \int_{0}^{x} p(t)k(t)dt. \]

If \( k(x) = 1 \) for all \( x \in \mathbb{R} \), we drop the subscript and write \( M \) and \( L \).

- (1) Show that \( M_{k(x)} \) and \( L_{k(x)} \) are linear transformations for all polynomials \( k(x) \). Calculate

\[ M_{x^2 + x^4}, L_{1 + x^3}. \]

(2) Calculate \( D : M_{k(x)} - M_{k(x)} \cdot D : P(\mathbb{R}) \rightarrow P(\mathbb{R}) \), where \( D \) is defined in Example 2 of Section 8.1. That is, find a formula for

\[ (D \cdot M_{k(x)} - M_{k(x)} \cdot D)(p(x)). \]

(3) Show that \( L : M_{k(x)} = L_{k(x)} : P(\mathbb{R}) \rightarrow P(\mathbb{R}) \).

2. Show that \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by

\[ T(x, y) = (ax + by, cx + dy), \]

\( a, b, c, d \) fixed real numbers, is a linear transformation and that \( T \) is an isomorphism if and only if \( ad \neq bc \).

3. Show that \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by

\[ T(x, y, z) = (a_1x + a_2y + a_3z, b_1x + b_2y + b_3z, c_1x + c_2y + c_3z) \]

is a linear transformation. Any linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) takes this form, where \( a_i, b_j, c_k \) are scalars.

4. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( T(x, y) = (x - y, 2x + y) \) and let \( S : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( S(x, y) = (y - 2x, x + y) \).

   - (1) Find \( T \cdot S(1, 0), T + S(1, 0), S \cdot T(1, 0), (2T - S)(1, 0), S^2(1, 0) \), and \( T^2(1, 0) \).
   - (2) Find \( T \cdot S(x, y) \) and \( S \cdot T(x, y) \).
   - (3) What are the vectors \( (x, y) \) satisfying \( T(x, y) = (1, 0) \)?

5. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( T(x, y) = (x, x) \). What are the kernel of \( T \) and the image of \( T \)?

6. Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be defined by \( T(x, y, z) = (x + y, y + z, x + z) \). Is \( T \) an isomorphism? Find \( T(1, 0, 0), T(0, 1, 0) \) and \( T(0, 0, 1) \).

7. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a linear transformation satisfying the condition

\[ T(x, y, z) = (0, 0, 0) \text{ whenever } 2x - y + z = 0. \]

Let \( T(0, 0, 1) = (1, 2, 3), T(1, 0, 0) \), and \( T(0, 1, 0) \). What is \( \text{dim}(\text{Im}(T)) \)?

8. Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the linear transformation given by the formula

\[ T(x, y, z) = (y + z, x + z, x + y) \]  

Show that \( T \) is an isomorphism and find an inverse for \( T \).

9. Let \( V \) be a finite-dimensional vector space over \( \mathbb{R} \), and \( E_1, \ldots, E_n \) a basis for \( V \). Define for each \( i, 1 \leq i \leq n \), a function

\[ E'_i : V' \rightarrow \mathbb{R} \]

by the formula

\[ E'_i(A) = a_i, \quad 1 \leq i \leq n, \]

where

\[ A = a_1E_1 + \cdots + a_nE_n. \]

- (1) Show that \( E'_1, \ldots, E'_n \) are linear transformations.
- (2) Find a basis for \( \text{ker}(E'_i) \).
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(3) Let \( S : V' \to \mathcal{L}(V, \mathbb{R}) \) be the linear extension of the map
\[
E_i \mapsto E_i', \quad i = 1, \ldots, n.
\]
Show that \( S \) is an isomorphism.

(4) Compute \( \dim_{\mathbb{R}} L(V', \mathbb{R}) \).

10. Let \( V' \) be a vector space and define \( V'' = L(V', \mathbb{R}) \). Define a linear transformation
\[
B : V' \to V'' \quad (= \mathcal{L}(V', \mathbb{R}) = \mathcal{L}(L(V', \mathbb{R}), \mathbb{R}))
\]
by
\[
B(A)(f) = f(A).
\]

(1) Show that \( B \) is a linear transformation.

(2) Show that \( \ker(B) = \{0\} \).

(3) Show that \( B \) is an isomorphism if \( V' \) is finite-dimensional.

(4) Show by an example that \( \text{Im}(B) \neq V'' \) when \( V' \) is not finite-dimensional.

11. Let \( E = \{E_1, \ldots, E_n\} \) be a basis for the finite-dimensional vector space \( V' \). Show that the linear extension construction gives an isomorphism
\[
L : \text{Fun}(E) \to V'
\]
by
\[
L(f) = f(E_1) \cdot E_1 + \cdots + f(E_n) \cdot E_n.
\]

12. Let \( E = \{E_1, \ldots, E_n\} \) be a basis for the finite-dimensional vector space \( V' \). Show that the map \( \text{Fun}(E) \to \mathcal{L}(V', \mathbb{R}) \) and the remarks following Theorem 8.5.6)

\[
C : V'' \to \text{Fun}(E)
\]
defined by
\[
C(T)(E_i) = T(E_i), \quad i = 1, \ldots, n,
\]
is a vector space isomorphism.

13. Define a function
\[
S : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})
\]
by
\[
S(p(x)) = p(x + 1).
\]
Determine whether \( S \) is a linear transformation.

14. Let \( V' \) and \( W \) be finite-dimensional vector spaces. Suppose \( E = \{E_1, \ldots, E_n\} \) is a basis for \( V' \) and \( \{F_1, \ldots, F_m\} \) a basis for \( W \). For each ordered pair \((i, j)\) show that the function
\[
M_{i,j}(A) = a_j F_i, \quad A = a_1 E_1 + \cdots + a_n E_n,
\]
is a linear transformation.

(1) Show that \( \{M_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\} \) is a basis for \( \mathcal{L}(V, W) \).

(2) Show that \( \dim(\mathcal{L}(V', W')) = nm \).

15. Let \( V' \) be a vector space and suppose that \( S, T : V' \to L(V, \mathbb{R}) \). Define
\[
L : V' \to \mathbb{R}^2
\]
by
\[
L(A) = (S(A), T(A)).
\]

(1) Show that \( L \) is a linear transformation.

(2) Show that \( \ker(L) \subset \ker(S) \cap \ker(T) \).

(3) Let \( V' \) be a vector space and define (see Theorem 8.5.6)
\[
C_i : L(V', \mathbb{R}) \to L(V', \mathbb{R}^2), \quad i = 1, 2, \quad C_i(T)(A) = \text{Hom}_{\mathbb{R}}(V, \mathbb{R}^2)
\]
by
\[
C_i(T)(A) = (T(A), 0), \quad C_i(T)(A) = (0, T(A)), \quad \text{Hom}_{\mathbb{R}}(V, \mathbb{R}^2)
\]

(1) Show that \( C_i \) is a linear transformation for \( i = 1, 2 \).

(2) Show that \( \ker(C_i) = \{0\} \), \( i = 1, 2 \).

(3) Show that \( \text{Im}(C_1) \cap \text{Im}(C_2) = \{0\} \).

(4) Let \( L_i = \text{Im}(C_i) \), \( i = 1, 2 \) and show that
\[
L(V', \mathbb{R}^2) = L_1 + L_2.
\]

(3) Let \( V' \) be a vector space over \( \mathbb{R} \). A complex structure on \( V' \) is a linear transformation
\[
J : V' \to V'
\]
such that for all \( A \) in \( V' \),
\[
J^2(A) = -A.
\]
(1) Show that
\[
J(x, y) = (-y, x)
\]
is a complex structure on \( \mathbb{R}^2 \).

(2) Show that
\[
J(x, y) = (y, -x)
\]
is a different complex structure on \( \mathbb{R}^2 \).
10.5 Exercises

1. For each of the following linear transformations of $\mathbb{R}^3$ to $\mathbb{R}^3$ calculate the matrix relative to the standard basis:
   (1) $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = (x + y + z, 0, 0)$.
   (2) $Q : \mathbb{R}^3 \to \mathbb{R}^3$ by $Q(x, y, z) = (x, x + y, x + y + z)$.
   (3) $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x, y, z) = (x + 2y + 3z, 2x + y, z - x)$.
   (4) $G : \mathbb{R}^3 \to \mathbb{R}^3$ by $G(x, y, z) = (y - z, x + y, z - 2x)$.

2. Calculate the matrices of the linear transformations
   
   $T \cdot Q : \mathbb{R}^3 \to \mathbb{R}^3$
   
   $F \cdot G : \mathbb{R}^3 \to \mathbb{R}^3$
   
   $Q \cdot F \cdot G : \mathbb{R}^3 \to \mathbb{R}^3$

   where $T$, $Q$, $F$, and $G$ are as in Exercise 1.

3. Suppose that the matrix of the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ is

   $S = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$

   relative to the standard basis of $\mathbb{R}^3$. Calculate $S(1, 2, 3)$.

4. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

   $P(x, y, z) = (x, y, 0)$.

   Calculate the matrix for $P$ and for $P^2$ relative to the standard basis of $\mathbb{R}^3$.

5. Let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations with matrices $A$ and $B$ relative to the standard basis of $\mathbb{R}^3$. What is the matrix of $S + T$ relative to this basis?

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear extension of

   $T(e_1) = (-1, 0, 2)$,
   $T(e_2) = (1, 1, -1)$,
   $T(e_3) = (1, -3, 4)$.

   Calculate the matrix of $T$ relative to the standard basis.
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7. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear extension of
   \[ T(1, 1) = (1, -1), \]
   \[ T(1, -2) = (2, 2). \]

   Calculate the matrix of the linear transformation $T$ relative to the standard basis.

8. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by
   \[ T(x, y, z) = (0, y, z). \]

   Calculate the matrix of $T$ relative to the standard basis of $\mathbb{R}^3$.

9. Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear transformation defined by
   \[ T(x, y, z, w) = (0, x, y, z). \]

   Calculate the matrix of $T$ relative to the standard basis. Calculate the matrices of $T^2$, $T^3$, $T^4$ relative to the standard basis.

10. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation whose matrix relative to the standard basis is
    \[
    \begin{bmatrix}
    1 & 2 & 2 \\
    0 & 1 & 1 \\
    2 & -1 & 0
    \end{bmatrix}
    \]

    (1) Calculate $T(E_1)$, $T(E_2)$, $T(E_3)$.
    (2) Let $r$ be a real number. Calculate the matrix of $rT$ relative to the standard basis.

11. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation whose matrix relative to the standard basis is
    \[
    \begin{bmatrix}
    1 & 1 \\
    1 & 2
    \end{bmatrix}
    \]

    (1) Calculate $T(E_1)$ and $T(E_2)$.
    (2) Find $A_1, A_2$ satisfying $T(A_i) = E_i, i = 1, 2$.

12. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation whose matrix relative to the standard basis is
    \[
    \begin{bmatrix}
    2 & 0 & -1 \\
    1 & 1 & 0 \\
    -1 & 1 & 1
    \end{bmatrix}
    \]

    (1) Calculate $T(E_1)$, $T(E_2)$, $T(E_3)$, and $T(1, 2, 3)$.

13. Let $A_1, A_2, A_3 \in \mathbb{R}^3$ and write
    \[ A_j = (a_{1,j}, a_{2,j}, a_{3,j}), \quad j = 1, 2, 3. \]

    Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear extension of the function $T(E_j) = A_j, j = 1, 2, 3$. Show that the matrix of $T$ relative to the standard basis of $\mathbb{R}^3$ is $(a_{i,j})$.

14. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation and $A = (a_{i,j})$ its matrix with respect to the standard basis of $\mathbb{R}^2$. Regard the columns of $A$ as vectors in $\mathbb{R}^2$. Call these three vectors $A_1, A_2, A_3$. Show that $\text{Im}(T) = \langle (A_1, A_2, A_3) \rangle$.

15. If $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the linear transformation with matrix
    \[
    \begin{bmatrix}
    0 & 1 & 1 \\
    1 & 0 & 1 \\
    1 & 1 & 0
    \end{bmatrix}
    \]

    with respect to the standard basis of $\mathbb{R}^3$, show that $T$ is an isomorphism.

16. Perform the following matrix computations:

    (a) $2 \begin{bmatrix} 3 & 1 & 4 & -4 \\ 2 & 0 & -1 & 2 \\ -3 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 7 \\ -1 & 2 & 0 & -4 \end{bmatrix}$

    (b) $2 \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} + 8 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

17. Perform the following matrix multiplications:

    (a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

    (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$

    (c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

    (d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

    (e) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$