A range test method for an inverse source problem

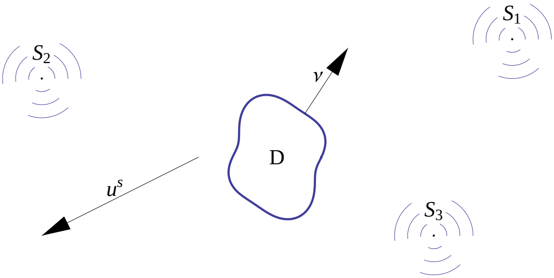
Rainer Kress
University of Göttingen

Joint work with:

Carlos Alves, Lisbon, and Pedro Serranho, Coimbra

Tainan, January 2011

Scattering from point sources



Incident Wave:

$$u^i(x) = \sum_{j=1}^{n} c_j \Phi(x, s_j)$$

Outline

1. The inverse problem
2. Iterative solution
3. Range test method
4. Tikhonov regularization
5. Examples

Range test methods

Idea: For nonlinear inverse scattering problem, design an ill-posed linear integral equation of the form

$$A\varphi = f_z \qquad \text{or} \qquad A_\Gamma \varphi = f$$

with a right hand side f depending on a point z or an operator A depending on a closed curve Γ.

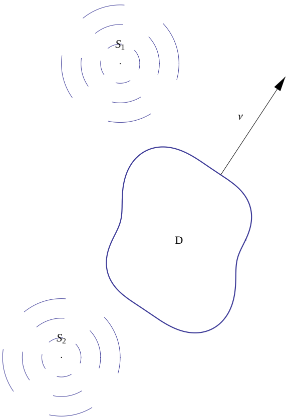
Idea: For nonlinear inverse scattering problem, design an ill-posed linear integral equation of the form

$$A\varphi = f_z \qquad \text{or} \qquad A_\Gamma \varphi = f$$

with a right hand side f depending on a point z or an operator A depending on a closed curve Γ. Via solvability of this equation decide on whether

- the point z belongs to the unknown scatterer or not
- or the scatterer lies in the interior of the curve Γ or not.

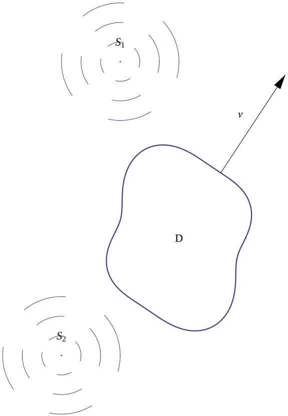
Scattering from point sources



Incident Wave:

$$u^i(x) = \sum_{j=1}^{n} c_j \Phi(x, s_j)$$

- D bounded smooth domain in $\mathbb{R}^2$
- $u^s \in C(\mathbb{R}^2 \setminus D) \cap C^2(\mathbb{R}^2 \setminus \overline{D})$



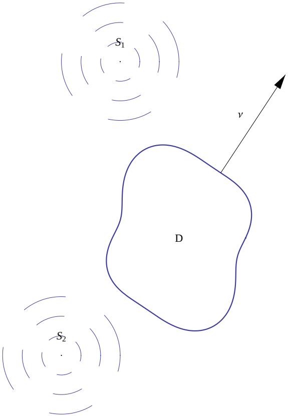
- D bounded smooth domain in $\mathbb{R}^2$
- $u^s \in C(\mathbb{R}^2 \setminus D) \cap C^2(\mathbb{R}^2 \setminus \overline{D})$

Helmholtz equation:

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}$$

Incident Wave:

$$u^i(x) = \sum_{j=1}^{n} c_j \Phi(x, s_j)$$



- D bounded smooth domain in $\mathbb{R}^2$
- $u^s \in C(\mathbb{R}^2 \setminus D) \cap C^2(\mathbb{R}^2 \setminus \overline{D})$

Helmholtz equation:

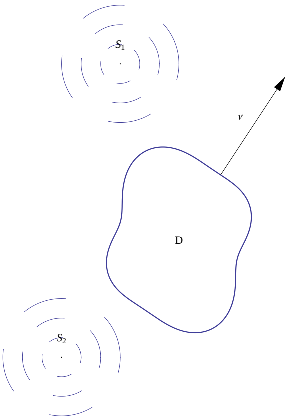
$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}$$

Boundary condition:

$$u^s = -u^i \quad \text{on } \partial D$$

Incident Wave:

$$u^i(x) = \sum_{j=1}^{n} c_j \Phi(x, s_j)$$



- D bounded smooth domain in $\mathbb{R}^2$
- $u^s \in C(\mathbb{R}^2 \backslash D) \cap C^2(\mathbb{R}^2 \backslash \overline{D})$

Helmholtz equation:

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D}$$

Boundary condition:

$$u^s = -u^i \quad \text{on } \partial D$$

Sommerfeld radiation condition:

$$\frac{\partial u^s}{\partial r} - iku^s = o\left(\frac{1}{\sqrt{r}}\right), \; r = |x| \to \infty$$

Incident Wave:

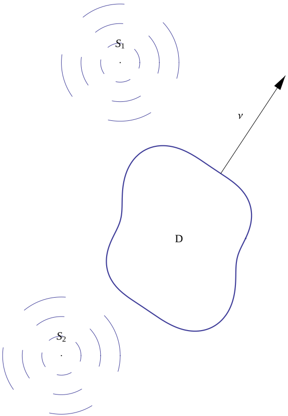
$$u^i(x) = \sum_{j=1}^{n} c_j \Phi(x, s_j)$$

Fundamental solution

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}\left(k|x-y|\right), \quad x \neq y$$

$$-\Delta\Phi(.,y) - k^2\Phi(.,y) = \delta_y$$

Scattering from point sources



- D bounded smooth domain in $\mathbb{R}^2$
- $u^s \in C(\mathbb{R}^2 \setminus D) \cap C^2(\mathbb{R}^2 \setminus \overline{D})$

Helmholtz equation ($u := u^i + u^s$):

$$-\Delta u - k^2 u = \sum_{j=1}^{n} c_j \delta_{s_j} \quad \text{in } \mathbb{R}^2 \setminus \overline{D}$$

Boundary condition:

$$u = 0 \quad \text{on } \partial D$$

Incident Wave:

$$u^i(x) = \sum_{j=1}^{n} c_j \Phi(x, s_j)$$

Sommerfeld radiation condition:

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right), \; r = |x| \to \infty$$

The inverse problem

$$\begin{cases} -\Delta u - k^2 u = \sum_{j=1}^{n} c_j \delta_{s_j} \quad \text{in } \mathbb{R}^2 \setminus \overline{D} \\ u = 0 \text{ on } \partial D \\ \dfrac{\partial u}{\partial r} - iku = o\left(\dfrac{1}{\sqrt{r}}\right), \quad r = |x| \to \infty \end{cases}$$

The inverse problem

$$\begin{cases} -\Delta u - k^2 u = \sum_{j=1}^{n} c_j \delta_{s_j} & \text{in } \mathbb{R}^2 \setminus \overline{D} \\ u = 0 \text{ on } \partial D \\ \dfrac{\partial u}{\partial r} - iku = o\left(\dfrac{1}{\sqrt{r}}\right), & r = |x| \to \infty \end{cases}$$

Inverse source problem

Given:

- the domain D
- the normal derivative $\dfrac{\partial u}{\partial \nu} = g$ on ∂D

Find:

- positions s_j and intensities c_j of the sources

Related Work

Ben Abda, Ben Hassen, Leblond, and Majoub 2008

El Badia and Ha-Duong 2000

Leblond, Paduret, Rigat, and Zghal 2008

Identifiability

The positions s_j and intensities c_j of the sources are uniquely determined by the normal derivative $\dfrac{\partial u}{\partial \nu} = g$ on ∂D.

(Follows from Holmgren's theorem.)

Inverse Problem

Identifiability

The positions s_j and intensities c_j of the sources are uniquely determined by the normal derivative $\dfrac{\partial u}{\partial \nu} = g$ on ∂D. (Follows from Holmgren's theorem.)

Non-linearity

The inverse problem is non-linear with respect to the positions s_j of the sources.

Inverse Problem

Identifiability

The positions s_j and intensities c_j of the sources are uniquely determined by the normal derivative $\dfrac{\partial u}{\partial \nu} = g$ on ∂D.
(Follows from Holmgren's theorem.)

Non-linearity

The inverse problem is non-linear with respect to the positions s_j of the sources.

Ill-posedness

The solution of the inverse problem does not depend continuously on the data.

Reciprocity relation

Total field:

$$\begin{cases} -\Delta u - k^2 u = \sum_{j=1}^{n} c_j \delta_{s_j} \quad \text{in } \mathbb{R}^2 \setminus \overline{D} \\ u = 0, \quad \dfrac{\partial u}{\partial \nu} = g \quad \text{on } \partial D \\ \dfrac{\partial u}{\partial r} - iku = o\left(\dfrac{1}{\sqrt{r}}\right), \quad r = |x| \to \infty \end{cases}$$

Reciprocity relation

Total field:

$$\begin{cases} -\Delta u - k^2 u = \sum_{j=1}^n c_j \delta_{s_j} \quad \text{in } \mathbb{R}^2 \setminus \overline{D} \\ u = 0, \quad \dfrac{\partial u}{\partial \nu} = g \quad \text{on } \partial D \\ \dfrac{\partial u}{\partial r} - iku = o\left(\dfrac{1}{\sqrt{r}}\right), \quad r = |x| \to \infty \end{cases}$$

Green's theorem:

For any solution v of the Helmholtz equation in $\mathbb{R}^2 \setminus \overline{D}$ satisfying the radiation condition we have

$$\sum_{j=1}^n c_j v(s_j) = \int_{\partial D} vg \, ds$$

Reciprocity relation

Reciprocity relation

For any radiating solution v to the Helmholtz equation in $\mathbb{R}^2 \setminus \overline{D}$ we have

$$\sum_{j=1}^{n} c_j v(s_j) = \int_{\partial D} vg \, ds$$

Reciprocity relation

Reciprocity relation

For any radiating solution v to the Helmholtz equation in $\mathbb{R}^2\setminus\overline{D}$ we have

$$\sum_{j=1}^{n} c_j v(s_j) = \int_{\partial D} vg\, ds$$

Choose test functions $v_1, \dots v_m$ with $m \geq 3n$ to obtain overdetermined system of equations

$$\sum_{j=1}^{n} c_j v_\ell(s_j) = \int_{\partial D} v_\ell g\, ds, \quad \ell = 1, \dots, m.$$

Reciprocity relation

For any radiating solution v to the Helmholtz equation in $\mathbb{R}^2 \setminus \overline{D}$ we have

$$\sum_{j=1}^{n} c_j v(s_j) = \int_{\partial D} vg \, ds$$

Choose test functions $v_1, \dots v_m$ with $m \geq 3n$ to obtain overdetermined system of equations

$$\sum_{j=1}^{n} c_j v_\ell(s_j) = \int_{\partial D} v_\ell g \, ds, \quad \ell = 1, \dots, m.$$

Solve iteratively via linearization, i.e., by Newton iterations.

Questions:

- How to choose the test functions v_ℓ?
- How to obtain an initial guess?

Initial guess via range test method

Denote by N the Neumann-to-Dirichlet operator for the exterior domain $\mathbb{R}^2\backslash\overline{D}$, that is,

$$N : \left.\frac{\partial v}{\partial \nu}\right|_{\partial D} \mapsto v|_{\partial D}\,,$$

for solutions v to Helmholtz equation in $\mathbb{R}^2\backslash\overline{D}$ satysfying the radiation condition.

Initial guess via range test method

Denote by N the Neumann-to-Dirichlet operator for the exterior domain $\mathbb{R}^2 \backslash \overline{D}$, that is,

$$N : \left. \frac{\partial v}{\partial \nu} \right|_{\partial D} \mapsto v|_{\partial D} \,,$$

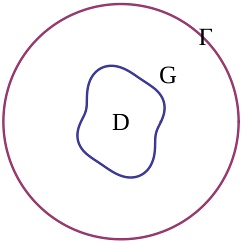
for solutions v to Helmholtz equation in $\mathbb{R}^2 \backslash \overline{D}$ satysfying the radiation condition.

Denote by u_0 be the solution to the exterior Neumann problem with boundary condition

$$\frac{\partial u_0}{\partial \nu} = g \quad \text{on } \partial D,$$

that is, $Ng = u_0|_{\partial D}$.

Initial guess via range test method

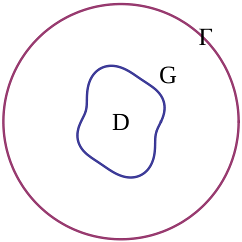


We can represent as a single-layer potential

$$u(x) - u_0(x) = \int_{\partial D} \Phi(x,y)\psi(y)\,ds(y) + \int_{\Gamma} \Phi(x,y)\varphi(y)\,ds(y), \quad x \in G,$$

if and only if $s_j \notin G$ for $j = 1, 2, \ldots, n$.

Initial guess via range test method



We can represent as a single-layer potential

$$u(x)-u_0(x) = \int_{\partial D} \Phi(x,y)\psi(y)ds(y) + \int_{\Gamma} \Phi(x,y)\varphi(y)ds(y), \quad x \in G,$$

if and only if $s_j \notin G$ for $j = 1, 2, \dots, n$.

Use boundary conditions $u - u_0 = -Ng$ and $\partial_\nu u - \partial_\nu u_0 = 0$ on ∂D to derive integral equation of the first kind for φ.

Initial guess via range test method

Define compact operators $V, W, A : L^2(\Gamma) \to L^2(\partial D)$ by

$$(V\varphi)(x) := \int_\Gamma \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \partial D,$$

and

$$(W\varphi)(x) := \int_\Gamma \frac{\partial \Phi(x, y)}{\partial \nu(x)}\, \varphi(y)\, ds(y), \quad x \in \partial D.$$

and $A := -V + NW$.

Theorem

Assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian neither for G nor for D. Then the ill-posed linear operator equation

$$A\varphi = Ng$$

is solvable for φ if and only if $s_j \notin G$ for $j = 1, 2, \ldots, n$.

Initial guess via range test method

Define compact operators $V, W, A : L^2(\Gamma) \to L^2(\partial D)$ by

$$(V\varphi)(x) := \int_\Gamma \Phi(x,y)\varphi(y)\,ds(y), \quad x \in \partial D,$$

and

$$(W\varphi)(x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(x)}\,\varphi(y)\,ds(y), \quad x \in \partial D.$$

and $A := -V + NW$.

Theorem

Assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian neither for G nor for D. Then the ill-posed linear operator equation

$$A\varphi = Ng$$

is solvable for φ if and only if $s_j \notin G$ for $j = 1, 2, \ldots, n$.

Question: How to check solvability numerically?

Tikohonov regularization

Approximate the ill-posed equation

$$A\varphi = f$$

by the well-posed equation

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$$

Tikohonov regularization

Approximate the ill-posed equation

$$A\varphi = f$$

by the well-posed equation

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$$

Assume that A and A^* are injective. Then the limit

$$\lim_{\alpha\to 0} \varphi_\alpha$$

exists if and only if the ill-posed equation is solvable.

Tikohonov regularization

Approximate the ill-posed equation

$$A\varphi = f$$

by the well-posed equation

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$$

Assume that A and A^* are injective. Then the limit

$$\lim_{\alpha\to 0} \varphi_\alpha$$

exists if and only if the ill-posed equation is solvable.

Check solvability:

- Perform Tikhonov regularization for a couple of small regularization parameters.
- If the solution φ_α changes only slightly, then the equation is considered solvable.
- Otherwise, the equation is considered unsolvable.

Injectivity and Dense Range

Theorem

The operator $A = -V + NW$ is injective provided k^2 is not a Dirichlet eigenvalue of the negative Laplacian for $D \cup G$.

Injectivity and Dense Range

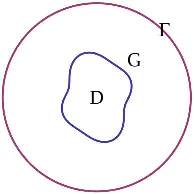
Theorem

The operator $A = -V + NW$ is injective provided k^2 is not a Dirichlet eigenvalue of the negative Laplacian for $D \cup G$.

Theorem

The operator $A := -V + NW$ has dense range provided k^2 is not a Dirichlet eigenvalue of the negative Laplacian for D.

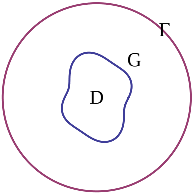
Check solvability



$$A\varphi = Ng$$

$$\alpha\varphi_\alpha + A^* A\varphi_\alpha = A^* Ng$$

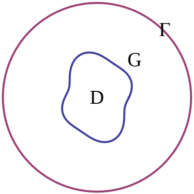
Check solvability



$$A\varphi = Ng$$
$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*Ng$$

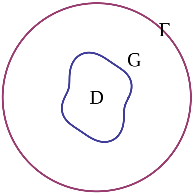
- Start with a circle Γ close to ∂D.

Check solvability



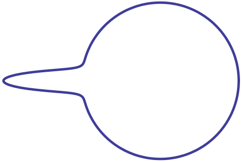
$$A\varphi = Ng$$
$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*Ng$$

- Start with a circle Γ close to ∂D.
- Enlarge the circle while the equation is solvable.

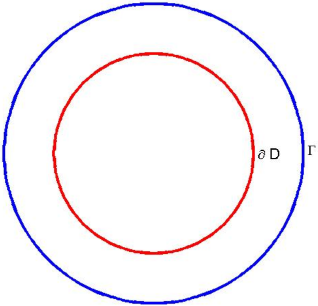


$$A\varphi = Ng$$
$$\alpha\varphi_\alpha + A^* A\varphi_\alpha = A^* Ng$$

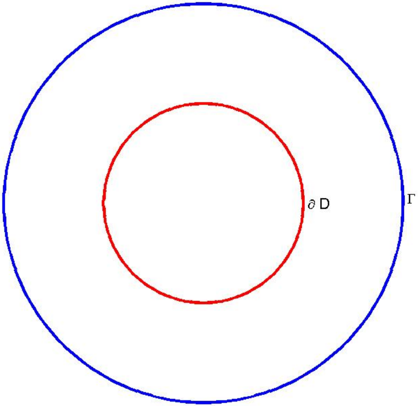
- Start with a circle Γ close to ∂D.
- Enlarge the circle while the equation is solvable.
- When it becomes unsolvable, use a circle with a bump for Γ and rotate it while testing solvability.



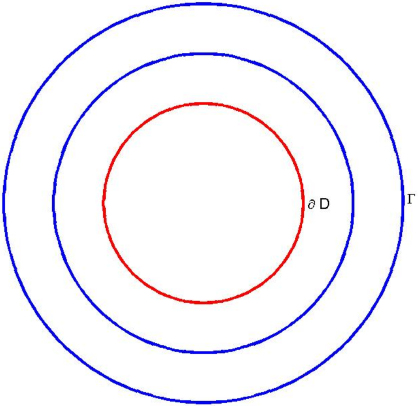
Check solvability



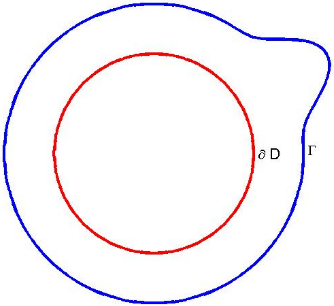
Check solvability



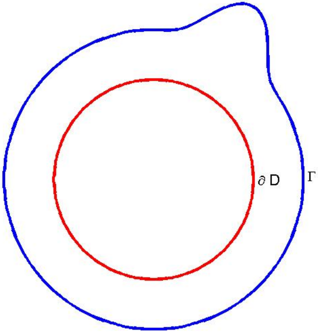
Check solvability



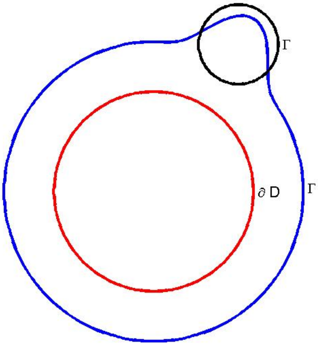
Check solvability



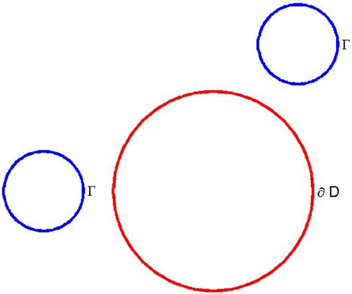
Check solvability



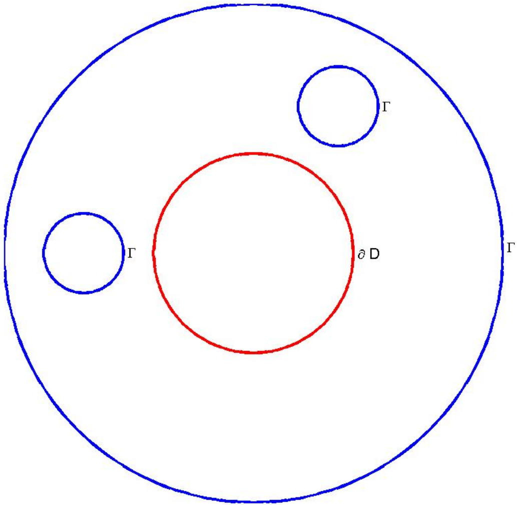
Check solvability



Check solvability



Check solvability



Integral equation of the second kind

$$\varphi(x) + \int_0^1 K(x,y)\varphi(y)\,dy = f(x), \quad 0 \le x \le 1$$

Integral equation of the second kind

$$\varphi(x) + \int_0^1 K(x,y)\varphi(y)\,dy = f(x), \quad 0 \le x \le 1$$

Approximate solution via trapezoidal rule:

$$\varphi_\ell + h \sum_{j=0}^{n}{}' K(\ell h, jh)\varphi_j = f(\ell h), \quad \ell = 0, 1, \ldots, n$$

Linear system for approximations $\varphi_\ell \approx \varphi(\ell h)$.

Integral equation of the second kind

$$\varphi(x) - \frac{1}{2} \int_0^1 (x+1)e^{-xy}\varphi(y)dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}$$

has solution $\varphi(x) = e^{-x}$

Integral equation of the second kind

$$\varphi(x) - \frac{1}{2}\int_0^1 (x+1)e^{-xy}\varphi(y)\,dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}$$

has solution $\varphi(x) = e^{-x}$

Trapezoidal rule

n	$x = 0$	$x = 0.5$	$x = 1$
4	-0.007146	-0.010816	-0.015479
8	-0.001788	-0.002711	-0.003882
16	-0.000447	-0.000678	-0.000971
32	-0.000112	-0.000170	-0.000243

Integral equation of the second kind

$$\varphi(x) - \frac{1}{2}\int_0^1 (x+1)e^{-xy}\varphi(y)\,dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}$$

has solution $\varphi(x) = e^{-x}$

Simpson rule

n	$x = 0$	$x = 0.5$	$x = 1$
4	-0.00006652	-0.00010905	-0.00021416
8	-0.00000422	-0.00000692	-0.00001366
16	-0.00000026	-0.00000043	-0.00000086
32	-0.00000002	-0.00000003	-0.00000005

Integral equation of the first kind

$$\int_0^1 K(x,y)\varphi(y)\,dy = f(x), \quad 0 \le x \le 1$$

Integral equation of the first kind

$$\int_0^1 K(x,y)\varphi(y)\,dy = f(x), \quad 0 \le x \le 1$$

Approximate solution via trapezoidal rule:

$$h\sum_{j=0}^{n}{}' K(\ell h, jh)\varphi_j = f(\ell h), \quad \ell = 0, 1, \dots, n$$

Linear system for approximations $\varphi_\ell \approx \varphi(\ell h)$.

Integral equation of the first kind

$$\int_0^1 (x+1)e^{-xy}\varphi(y)dy = 1 - e^{-(x+1)}$$

has solution $\varphi(x) = e^{-x}$

Integral equation of the first kind

$$\int_0^1 (x+1)e^{-xy}\varphi(y)dy = 1 - e^{-(x+1)}$$

has solution $\varphi(x) = e^{-x}$

Trapezoidal rule

n	$x = 0$	$x = 0.5$	$x = 1$
4	0.4057	0.3705	0.1704
8	-4.5989	14.6094	-4.4770
16	-8.5957	2.2626	-153.4805
32	3.8965	-32.2907	22.5570
64	-88.6474	-6.4484	-182.6745

Integral equation of the first kind

$$\int_0^1 (x+1)e^{-xy}\varphi(y)dy = 1 - e^{-(x+1)}$$

has solution $\varphi(x) = e^{-x}$

Simpson rule

n	$x = 0$	$x = 0.5$	$x = 1$
4	0.0997	0.2176	0.0566
8	-0.5463	6.0868	-1.7274
16	-15.4796	50.5015	-53.8837
32	24.5929	-24.1767	67.9655
64	23.7868	-17.5992	419.4284

Tikhonov regularization

$$\int_0^1 (x+1)e^{-xy}\varphi(y)dy = 1 - e^{-(x+1)} \text{ has solution } \varphi(x) = e^{-x}$$

$\alpha = 10^{-8}$	n	$x = 0$	$x = 0.5$	$x = 1$
	16	-0.000359	-0.000002	0.000209
	32	-0.000193	-0.000014	0.000149
	64	-0.000183	0.000015	0.000145
	128	-0.000182	-0.000015	0.000145

$\alpha = 10^{-10}$	n	$x = 0$	$x = 0.5$	$x = 1$
	16	-0.000455	-0.000022	0.000277
	32	-0.000042	-0.000001	0.000017
	64	-0.000011	-0.000000	-0.000001
	128	-0.000008	-0.000002	0.000002

Tikhonov regularization

$$\int_0^1 (x+1)e^{-xy}\varphi(y)dy = 1 - e^{-(x+1)} \text{ has solution } \varphi(x) = e^{-x}$$

$\alpha = 10^{-8}$	n	$x = 0$	$x = 0.5$	$x = 1$
	32	0.999806	0.606516	0.368028
	64	0.999816	0.606515	0.368024
	128	0.999817	0.606515	0.368024

$\alpha = 10^{-10}$	n	$x = 0$	$x = 0.5$	$x = 1$
	32	0.999957	0.606528	0.367896
	64	0.999988	0.606529	0.367878
	128	0.999991	0.606527	0.367881

Tikhonov regularization

$$\int_0^1 (x+1)e^{-xy}\varphi(y)dy = \frac{1}{2} - \left|\frac{1}{2} - x\right| \text{ has no solution}$$

$\alpha = 10^{-8}$	n	$x = 0$	$x = 0.5$	$x = 1$
	32	12.3108	61.1597	-178.454
	64	12.2942	61.1537	-178.438
	128	12.2931	61.1533	-178.437

$\alpha = 10^{-10}$	n	$x = 0$	$x = 0.5$	$x = 1$
	32	1180.66	419.818	514.502
	64	1181.06	419.593	513.926
	128	1181.09	419.579	513.891

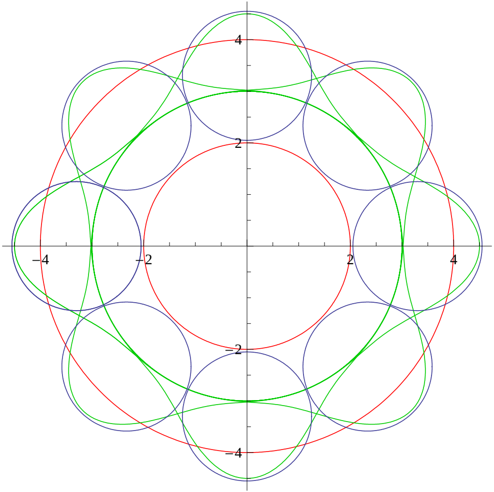
Tikhonov regularization

$$\int_0^1 (x+1)e^{-xy}\varphi(y)dy = 1 \qquad ???$$

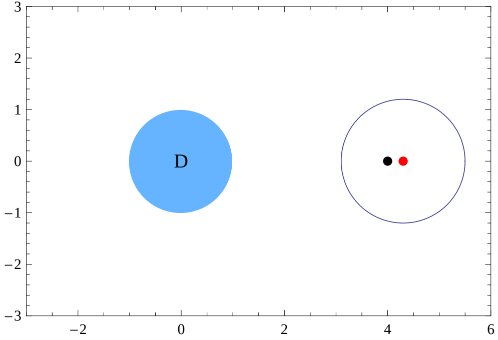
$\alpha = 10^{-8}$	n	$x = 0$	$x = 0.5$	$x = 1$
	32	-24.1059	-8.0327	45.9362
	64	-24.1052	-8.0314	45.9311
	128	-24.1052	-8.0314	45.9308

$\alpha = 10^{-10}$	n	$x = 0$	$x = 0.5$	$x = 1$
	32	-84.0948	-10.3093	131.5763
	64	-84.1591	-10.3218	131.5725
	128	-84.1632	-10.3226	131.5722

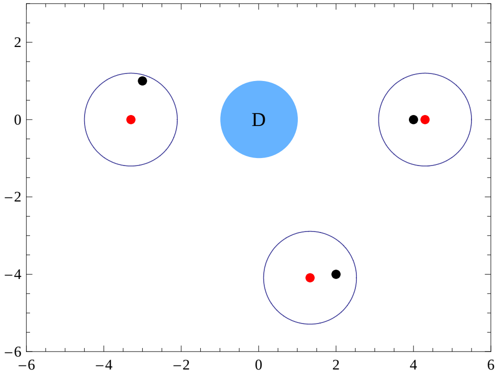
Example for inverse source problem



Example for inverse source problem



Example for inverse source problem



Reference

Alves, C., Kress, R. and Serranho, P.
Iterative and range test method for an inverse source problem for acoustic waves.
Inverse Problems **25**, 055055 (2009)