Using Geometric Inequality to Prove the Convergence of Nonlocal Flows

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THE PROBLEM

We are concerned with the following general "nonlocal" curvature flow of smooth convex closed curves:

$$\begin{cases} \frac{\partial \mathbf{X}}{\partial t}(\varphi, t) = [F(k(\varphi, t)) - \lambda(t)]\, \mathbf{N}_{in}(\varphi, t) \\ \mathbf{X}(\varphi, 0) = \mathbf{X}_0(\varphi), \quad \varphi \in S^1, \end{cases} \qquad (*)$$

which is a parabolic IVP. Here:

- $X_0(\varphi) : S^1 \to \gamma_0$ is a parametrization of the initial (convex) curve γ_0.
- $k(\varphi, t) =$ curvature of $\gamma_t = \gamma(\cdot, t)$ (parametrized by $X(\varphi, t) : S^1 \times [0, T) \to R^2$).
- $N_{in}(\varphi, t) =$ inward normal of γ_t.
- $F'(z) > 0$ for all $z \in$ domain of F (parabolic condition).
- $\lambda(t) =$ a function of time, which may depend on global quantities, say $L(t)$ or $A(t)$ of γ_t or others. If $\lambda(t)$ depends on γ_t, then it is not known beforehand.

We note the following:

- Many interesting physical models are nonlocal in nature.
- If γ_0 is not convex, then it may develop self-intersections. This will make (*) uncontrollable. Also, the $1/k$-type flows (see below) will be undefined.
- The RHS of (*) has no tangential component because it has NO essential effect at all.
- (*) is a 2×2 nonlinear degenerate parabolic system. We can overcome the degeneracy by looking at geometric quantities like "curvature" or "support function" to guarantee the existence of a smooth solution for short time.
- The goal is to study the asymptotic behavior of the flow.

Basic evolution formulas

Let $X(\varphi, t) : S^1 \times [0, T) \to R^2$ be a family of smooth time-dependent simple closed curves with time variation

$$\frac{\partial \mathbf{X}}{\partial t}(\varphi, t) = \mathbf{W}(\varphi, t) \in \mathbb{R}^2,$$

then its length $L(t)$ and enclosed area $A(t)$ satisfy the following:

$$\frac{dL}{dt}(t) = -\int_{\gamma_t} \langle \mathbf{W},\ k\mathbf{N}_{in} \rangle \, ds, \qquad \frac{dA}{dt}(t) = -\int_{\gamma_t} \langle \mathbf{W},\ \mathbf{N}_{in} \rangle \, ds \qquad (1)$$

where s is the arc length parameter of the curve $X(\varphi, t)$. Moreover,

$$\frac{d}{dt}\left[L^2(t) - 4\pi A(t)\right] = -2L(t) \int_{\gamma_t} \left\langle \mathbf{W},\ \left(k - \frac{2\pi}{L(t)}\right) \mathbf{N}_{in} \right\rangle ds \qquad (2)$$

$$\frac{d}{dt}\left[\frac{L^2(t)}{4\pi A(t)}\right] = -\frac{L(t)}{2\pi A(t)} \int_{\gamma_t} \left\langle \mathbf{W},\ \left(k - \frac{L(t)}{2A(t)}\right) \mathbf{N}_{in} \right\rangle ds \qquad (3)$$

Remark

By (1), (2), and (3), one can talk about the "gradient flow" of the length functional, the area functional (under suitable function space setting), or others.

1. $\mathbf{W}(\varphi, t) = k(\varphi, t)\,\mathbf{N}_{in}(\varphi, t) \quad (\textit{CSF; the gradient flow of } L).$
2. $\mathbf{W}(\varphi, t) = \mathbf{N}_{in}(\varphi, t) \quad \left(\begin{array}{l} \textit{the gradient flow of } A, \\ \textit{the unit speed inward normal flow} \end{array}\right).$
3. $\mathbf{W}(\varphi, t) = \left(k(\varphi, t) - \frac{2\pi}{L(t)}\right)\mathbf{N}_{in}(\varphi, t) \quad \left(\textit{gradient flow of } L^2 - 4\pi A\right).$
4. $\mathbf{W}(\varphi, t) = \left(k(\varphi, t) - \frac{L(t)}{2A(t)}\right)\mathbf{N}_{in}(\varphi, t) \quad \left(\textit{gradient flow of } L^2/4\pi A\right).$

Also its curvature $k(\varphi, t)$ satisfies

$$\frac{\partial k}{\partial t}(\varphi, t) = \left\langle \frac{\partial^2 \mathbf{W}}{\partial s^2},\ \mathbf{N}_{in} \right\rangle - 2k \left\langle \frac{\partial \mathbf{W}}{\partial s},\ \mathbf{T} \right\rangle \tag{4}$$

where $T = T(\varphi, t)$ is the unit tangent vector of $X(\varphi, t)$.

Note that here the operator $\partial/\partial s = \left|\mathbf{X}_\varphi(\varphi, t)\right|^{-1} \partial/\partial\varphi$ is time-dependent, which is usually not preferable.

Remark

When the curve has self-intersections, the above formulas for dL/dt and $\partial k/\partial t$ remain correct but not for dA/dt.

Some examples of nonlocal flow

Recall that γ_0 is convex with $k > 0$ everywhere.

- For k-type flows, there are:

$$\begin{cases} (I).\ F(k) - \lambda(t) = k - \dfrac{2\pi}{L(t)} & \left(\begin{array}{l} A\text{-preserving,} \\ \text{gradient flow of } L^2 - 4\pi A \end{array} \right) \\ (II).\ F(k) - \lambda(t) = k - \dfrac{L(t)}{2A(t)} & \left(\begin{array}{l} L\text{-decreasing, } A\text{-increasing,} \\ \text{gradient flow of } L^2/4\pi A \end{array} \right) \\ (III).\ F(k) - \lambda(t) = k - \dfrac{1}{2\pi}\displaystyle\int_{\gamma_t} k^2 ds & (L\text{-preserving}), \end{cases}$$

studied by Gage (1986), Jiang-Pan (2008) and Ma-Zhu (2008) respectively.

- For $1/k$-type flows, there are:

$$\begin{cases} (IV).\ F(k)-\lambda(t)=\frac{1}{L(t)}\int_{\gamma_t}\frac{1}{k}ds-\frac{1}{k} \quad (A\text{-preserving}) \\ (V).\ F(k)-\lambda(t)=\frac{L(t)}{2\pi}-\frac{1}{k} \quad (L\text{-preserving}), \\ (VI).\ F(k)-\lambda(t)=\frac{2A(t)}{L(t)}-\frac{1}{k} \quad (\text{the dual flow of } (II)), \end{cases}$$

studied by Ma-Cheng (2009), Pan-Yang (2008), and Lin-Tsai (2010) respectively.

A dual relation

- There is a dual relation between k-type flows and $1/k$-type flows.

For k-type flows with curvature speed $(k - p(t)) N_{in}$, we have

$$\frac{dL}{dt}(t) = -\int_{\gamma_t} k^2 ds + 2\pi p(t), \qquad \frac{dA}{dt}(t) = -2\pi + p(t) L(t)$$

and for $1/k$-type flows with curvature speed $(q(t) - 1/k) N_{in}$, we have

$$\frac{dL}{dt}(t) = -2\pi q(t) + L(t), \qquad \frac{dA}{dt}(t) = -q(t) L(t) + \int_{\gamma_t} \frac{1}{k} ds.$$

When $p(t) = 1/q(t)$, there is a "dual relation" between the above two, i.e.,

$$\frac{1}{q(t)} \frac{dL}{dt}(t) \text{ (for } 1/k\text{-type flows)} = \frac{dA}{dt}(t) \text{ (for } k\text{-type flows).}$$

Hence in the above, flows (I) and (V) are dual.

- Motivated by the dual relation, we (Lin-T. (2010)) study the following:

$$(VI).\ F(k) - \lambda(t) = \frac{2A(t)}{L(t)} - \frac{1}{k} \quad \text{(L-increasing, A-increasing)}$$

  which is dual to the flow (II).

- Furthermore, we study the existence of a general linear nonlocal curvature flow

$$\frac{\partial \mathbf{X}}{\partial t}(\varphi, t) = \left(H(L(t), A(t)) - \frac{1}{k(\varphi, t)} \right) \mathbf{N}_{in}(\varphi, t), \qquad (5)$$

where $H(p, q) : (0, \infty) \times (0, \infty) \rightarrow R$ is a given (but arbitrary) smooth function of two variables.

Short time existence of the solution

Theorem

Let γ_0 be smooth and convex. Then for each of the flows (I)-(VI) there exists a unique convex smooth solution γ_t on $S^1 \times [0, T)$ for short time $T > 0$.

Proof

- Uniqueness: The uniqueness is due to the maximum principle.
- Existence:
  For k-type flows, consider the evolution equation of the curvature in terms of normal angle θ:

$$\frac{\partial k}{\partial t}(\theta, t) = k^2(\theta, t)\left[k_{\theta\theta}(\theta, t) + k(\theta, t) - \lambda(t)\right] \tag{6}$$

  and one can use the argument as in Section 2 of Gage-Hamilton (1986) to prove the short time existence.

For $1/k$-type flows, one can give a direction proof since, using the "support function" $u(\theta, t)$, where θ is the outward normal angle, it gives rise to a "linear" equation

$$\frac{\partial u}{\partial t}(\theta, t) = u_{\theta\theta}(\theta, t) + u(\theta, t) - \lambda(t) \quad \text{on } S^1 \times [0, T). \tag{7}$$

Remark

The support function $u(\theta, t)$ of a convex curve γ_t is defined by

$$u(\theta, t) = \langle X(\theta, t), \ (\cos\theta, \sin\theta) \rangle, \quad \theta \in S^1 \tag{8}$$

where $X(\theta, t)$ is the position vector of the unique point γ_t with outward normal $N_{out} = (\cos\theta, \sin\theta)$. Using $u(\theta, t)$, we have

$$k(\theta, t) = \frac{1}{u_{\theta\theta}(\theta, t) + u(\theta, t)},$$

$$L(t) = \int_0^{2\pi} u(\theta, t)\, d\theta, \qquad A(t) = \frac{1}{2}\int_0^{2\pi} u(\theta, t)\left[u_{\theta\theta}(\theta, t) + u(\theta, t)\right] d\theta.$$

Remark (short time existence of a general linear nonlocal curvature flow)

In fact, for a general linear nonlocal curvature flow (5)*, this PDE problem can be resolved using* **an ODE method***, together with the help of representation formula for solutions to a linear heat equation. Consider*

$$\frac{\partial u}{\partial t}(\theta, t) = u_{\theta\theta}(\theta, t) + u(\theta, t) - H(L(t), A(t)). \tag{9}$$

Since

$$\frac{dL}{dt}(t) = L(t) - 2\pi H(L(t), A(t)),$$

let $w(\theta, t) = u(\theta, t) - L(t)/2\pi$ *and it satisfies the linear heat equation*

$$\frac{\partial w}{\partial t}(\theta, t) = w_{\theta\theta} + w$$

and thus by the representation formula we have

$$u(\theta, t) - \frac{L(t)}{2\pi} = \frac{e^t}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(\theta-\xi)^2}{4t}} \left[u(\xi, 0) - \frac{L(0)}{2\pi} \right] d\xi. \tag{10}$$

The evolution of $L(t)$ satifies the ODE

$$\begin{cases} \frac{dL}{dt}(t) = L(t) - 2\pi H\left(L(t), \frac{L^2(t)}{4\pi} + E(t) - \frac{L^2(0)}{4\pi}e^{2t}\right) \\ L(0) = \text{length of } \gamma_0 > 0, \end{cases} \tag{11}$$

where $E(t)$ is a time function determined by γ_0, given by

$$E(t) = \frac{1}{2}\int_0^{2\pi}\left\{ \begin{array}{l} \left(\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4t}} u(\theta-\xi,0)\, d\xi\right) \times \\ \left(\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4t}} \left[u_{\theta\theta}(\theta-\xi,0) + u(\theta-\xi,0)\right] d\xi\right) \end{array} \right\} d\theta$$

and $u(\theta,0)$ is the support function of γ_0.
The ODE (11) is now self-contained and by standard ODE theory there is a unique positive solution defined on interval $[0, T^*)$ for some $T^* > 0$. Thus we have short time existence of the solutions for (5).

Some useful inequalities related to L and A

To obtain the long time convergence of the flow, we need to rely on some useful geometric inequalities:

- General inequalities:

  Hölder inequality, Jensen inequality, Poincaré inequality, etc.

- Classical isoperimetric inequality:

$$\frac{2A}{L} \leq \sqrt{\frac{A}{\pi}} \leq \frac{L}{2\pi},$$

  where " $=$ " holds if and only if the curve is a circle.

- Andrews inequality:
  Let M be a compact Riemannian manifold with a volume form $d\mu$, and let ξ be a continuous function on M. Then for any **increasing** continuous function $F : R \to R$, we have

$$\int_M \xi d\mu \int_M F(\xi)\, d\mu \leq \int_M d\mu \int_M \xi F(\xi)\, d\mu.$$

  If F is strictly increasing, then equality holds if and only if ξ is a constant function on M.

Remark

In case $M = \gamma$ is a **convex** *closed curve with volume form ds, where s is arc length parameter, then the above becomes*

$$\int_\gamma \xi ds \int_\gamma F(\xi)\, ds \leq \int_\gamma ds \int_\gamma \xi F(\xi)\, ds, \tag{12}$$

where $\xi : \gamma \to \mathbb{R}$ is a continuous function.

- Gage inequality:
  For any convex closed curve γ, we have

$$\int_\gamma k^2(s)\,ds \geq \frac{\pi L}{A}, \quad \left(\text{note that } \int_\gamma k(s)\,ds = 2\pi\right)$$

  where s is arc length parameter. The equality holds if and only if γ is a circle.

- Curvature inequalities:

$$\int_0^{2\pi} k^q(\theta)\,d\theta \leq \frac{2A}{L}\int_0^{2\pi} k^{q+1}(\theta)\,d\theta, \quad q \geq 0 \text{ a constant},$$

  where θ is the outward normal angle of γ and the equality holds if and only if γ is a circle;
  or

$$\frac{L}{2\pi}\int_0^{2\pi} \frac{1}{k^q(\theta)}d\theta \leq \int_0^{2\pi} \frac{1}{k^{q+1}(\theta)}d\theta, \quad q \geq 0 \text{ a constant},$$

  where for $q > 0$ the equality holds if and only if γ is a circle.

- Green-Osher inequality:
  Let $Q(z) : (0, \infty) \to R$ be a function with $Q''(z) \geq 0$ everywhere. Then for convex γ with outward normal angle θ and curvature $k(\theta)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} Q\left(\frac{1}{k(\theta)}\right) d\theta \geq \frac{1}{2}\left[Q\left(\frac{L-\sqrt{L^2-4\pi A}}{2\pi}\right) + Q\left(\frac{L+\sqrt{L^2-4\pi A}}{2\pi}\right)\right]. \quad (13)$$

  - Taking $Q(z) = \frac{1}{z}$, we get Gage's inequality

$$\int_\gamma k^2(s)\, ds = \int_0^{2\pi} k(\theta)\, d\theta \geq \frac{\pi L}{A}.$$

  - Taking $Q(z) = z^2$, we have

$$\int_\gamma \frac{1}{k(s)} ds = \int_0^{2\pi} \frac{1}{k^2(\theta)} d\theta \geq \frac{L^2 - 2\pi A}{\pi}. \quad (14)$$

- Bonnesen inequality:
  For any convex closed curve γ, we have

$$L^2 - 4\pi A \geq \pi^2 \left(R_{out} - r_{in}\right)^2 \quad (15)$$

$$\frac{L^2}{4\pi A} - 1 \geq \pi \left(1 - \frac{r_{in}}{R_{out}}\right)^2 \quad (16)$$

where r_{in} and R_{out} are the inradius and circumradius of γ respectively.

Remark

(i) For a family of embedded closed curves γ_t, if $L^2(t)/4\pi A(t) \to 1$ as $t \to T$, then $r_{in}(t)/R_{out}(t) \to 1$ as $t \to T$. Hence γ_t evolves to become more and more circular and we say that γ_t converges to a circle in C_0 convergence.
(ii) As $A(t)$ is bounded away from zero, $L^2(t) - 4\pi A(t) \to 0$ iff $L^2(t)/4\pi A(t) \to 1$.

The decreasing of isoperimetric difference

Lemma

Assume $F' > 0$ everywhere (parabolic condition) and consider the flow

$$\frac{\partial \mathbf{X}}{\partial t}(\varphi, t) = \left[F\left(k\left(\varphi, t\right)\right) - \lambda\left(t\right)\right] \mathbf{N}_{in}\left(\varphi, t\right) \tag{17}$$

where $\lambda(t)$ is a time function which depends on L or A of γ_t. As long as $\gamma(\cdot, t)$ stays embedded on $[0, T)$, then the isoperimetric difference $L^2(t) - 4\pi A(t)$ for $\gamma(\cdot, t)$ is decreasing on $[0, T)$.

Proof The idea is to use Andrews's inequality. Compute

$$\begin{aligned} & \frac{d}{dt}\left[L^2(t) - 4\pi A(t)\right] \\ = \;& 2\left[\int_{\gamma_t} k ds \int_{\gamma_t} F(k)\, ds - \int_{\gamma_t} ds \int_{\gamma_t} kF(k)\, ds\right] \leq 0. \end{aligned}$$

Corollary

If flow (17) is A-preserving, then it is L-decreasing.
If it is L-preserving, then it is A-increasing.
In particular, if it preserves either A or L, then $L^2/4\pi A$ is decreasing.

Theorem

All flows (I)-(VI) are both $L^2 - 4\pi A$ and $L^2/4\pi A$ decreasing.

Theorem

(roughly) For flows (I)-(VI) we have smooth convex solution γ_t defined on $S^1 \times [0, \infty)$ and γ_t converges in C^∞ to round circle.

Andrews's affine $k^{1/3}$ flow; Blaschke's Theorem

For the curve contracting flow (γ_0 is convex)

$$\begin{cases} \dfrac{\partial \mathbf{X}}{\partial t}(\varphi, t) = k^{\alpha}(\varphi, t) \cdot \mathbf{N}_{in}(\varphi, t), \quad \alpha > 0 \text{ a constant} \\ \mathbf{X}(\varphi, 0) = \mathbf{X}_0(\varphi), \quad \varphi \in S^1, \end{cases}$$

Ben Andrews has proved the following:

Theorem

For any $\alpha > 0$, the curve γ_t contracts to a point in finite time.
(i) If $0 < \alpha < 1/3$, then for generic initial data there is no limit of the curves rescaled about the final point ($\limsup_{t \to T_{\max}} L^2/4\pi A = \infty$); and the exceptional ones where the isoperimetric ratio remains bounded converge to homothetic solutions, which have been classified.

(ii) For $\alpha > 1/3$, the rescaled solutions converge to circles and for $\alpha = 1/3$, they converge to ellipses.

A flow problem

Motivated by Andrews's theorem, it is interesting to consider a k^{α}-type nonlocal flow of the form:

$$\frac{\partial \mathbf{X}}{\partial t}(\varphi, t) = \left[k^{\alpha}(\varphi, t) - \lambda(t)\right] \mathbf{N}_{in}(\varphi, t) \tag{18}$$

where $\alpha > 0$ is a constant and $\lambda(t)$ is chosen by

$$\lambda(t) = \frac{1}{L(t)} \int_{\gamma_t} k^{\alpha} ds \quad (A\text{-preserving})$$

or

$$\lambda(t) = \frac{1}{2\pi} \int_{\gamma_t} k^{\alpha+1} ds \quad (L\text{-preserving}).$$

Consider a $1/k^{\alpha}$-type nonlocal flow of the flow

$$\frac{\partial \mathbf{X}}{\partial t}(\varphi, t) = \left[-\frac{1}{k^{\alpha}(\varphi, t)} - \lambda(t)\right] \mathbf{N}_{in}(\varphi, t) \tag{19}$$

where $\alpha > 0$ is a constant and $\lambda(t)$ is chosen by

$$\lambda(t) = -\frac{1}{L(t)} \int_{\gamma_t} \frac{1}{k^{\alpha}} ds \quad (A\text{-preserving})$$

or

$$\lambda(t) = \frac{-1}{2\pi} \int_{\gamma_t} \frac{1}{k^{\alpha-1}} ds \quad (L\text{-preserving}).$$

Applying Andrews's inequality, Gage's inequality and Green-Osher's inequality, we have the following (here the variable θ is the outward normal angle of γ):

1. $$\frac{2A}{L} \leq \sqrt{\frac{A}{\pi}} \leq \frac{L}{2\pi}. \tag{20}$$

2. $$\int_0^{2\pi} k^{p-1} d\theta \leq \frac{L}{2\pi} \int_0^{2\pi} k^p d\theta \quad \text{for any} \quad p \geq 0. \tag{21}$$

3. $$\int_0^{2\pi} k^p (\theta) \, d\theta \leq \frac{2A}{L} \int_0^{2\pi} k^{p+1} (\theta) \, d\theta, \quad p \geq 0. \tag{22}$$

4. $$\frac{L}{2\pi} \int_0^{2\pi} \frac{1}{k^p (\theta)} d\theta \leq \int_0^{2\pi} \frac{1}{k^{p+1} (\theta)} d\theta, \quad p \geq 0. \tag{23}$$

5. $$\frac{L^2 - 2\pi A}{L\pi} \int_0^{2\pi} \frac{1}{k^p (\theta)} d\theta \leq \int_0^{2\pi} \frac{1}{k^{p+1} (\theta)} d\theta, \quad p \geq 1. \tag{24}$$

Area-preserving k^α flow

Theorem

Assume $\alpha \geq 1$ and the area-preserving flow (18) has a smooth and convex solution defined on $S^1 \times [0, \infty)$. Then the isoperimetric difference of γ_t:

$$L^2(t) - 4\pi A(t) \searrow 0$$

exponentially as $t \to \infty$. In particular, by the Bonnesen's inequality, γ_t converges to a round circle with radius $\sqrt{\frac{A(0)}{\pi}}$.

Proof Compute

$$\frac{d}{dt}\left(L^2(t) - 4\pi A(t)\right) = -2L(t)\int_{\gamma_t}\left(k^\alpha - \lambda(t)\right) k ds$$

$$= 4\pi \int_0^{2\pi} k^{\alpha-1}(\theta, t)\, d\theta - 2L(t)\int_0^{2\pi} k^\alpha(\theta, t)\, d\theta \leq 0, \quad A(t) = A(0) \tag{25}$$

due to (21).

Since $\alpha \geq 1$, (22) gives

$$\int_0^{2\pi} k^{\alpha-1}(\theta,t)\,d\theta \leq \frac{2A(t)}{L(t)} \int_0^{2\pi} k^{\alpha}(\theta,t)\,d\theta, \quad \alpha \geq 1 \tag{26}$$

and then

$$\begin{aligned}
&4\pi \int_0^{2\pi} k^{\alpha-1}(\theta,t)\,d\theta - 2L(t)\int_0^{2\pi} k^{\alpha}(\theta,t)\,d\theta \\
&\leq \frac{-2}{L(t)}\left(L^2(t) - 4\pi A(t)\right)\int_0^{2\pi} k^{\alpha}(\theta,t)\,d\theta \\
&\leq \frac{-2}{L(0)}\left(L^2(t) - 4\pi A(t)\right)\int_0^{2\pi} k^{\alpha}(\theta,t)\,d\theta,
\end{aligned}$$

where by Gage's inequality and Hölder inequality, we also have

$$\frac{2\pi\sqrt{\pi}}{\sqrt{A(0)}} \leq \frac{\pi L(t)}{A(t)} \leq \int_0^{2\pi} k(\theta,t)\,d\theta \leq \left(\int_0^{2\pi} k^{\alpha}(\theta,t)\,d\theta\right)^{\frac{1}{\alpha}} (2\pi)^{1-\frac{1}{\alpha}}, \quad \alpha \geq 1.$$

Therefore there exists a positive constant $C > 0$ independent of time (C depends on α and γ_0) such that

$$\int_0^{2\pi} k^{\alpha}(\theta, t)\, d\theta \geq C \quad \text{for all} \quad t \in [0, \infty)$$

and then

$$\frac{d}{dt}\left(L^2(t) - 4\pi A(t)\right) \leq -\frac{2C}{L(0)}\left(L^2(t) - 4\pi A(t)\right) \tag{27}$$

for all time. Hence $L^2(t) - 4\pi A(t)$ decreases and decays to zero exponentially as $t \longrightarrow \infty$.

Finally, by the classical **Bonnesen inequality**

$$L^2(t) - 4\pi A(t) \geq \pi^2 \left(R(t) - r(t)\right)^2, \tag{28}$$

where $r(t)$ is the *inradius* of γ_t and $R(t)$ is the *circumradius* of γ_t, γ_t must converge to a round circle with

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty} r(t) = \sqrt{\frac{A(0)}{\pi}}.$$

Area-preserving $1/k^{\alpha}$ flow

Theorem

Assume $\alpha \geq 1$ *andthe area-preserving flow* (19) *has a smooth and convex solution defined on* $S^1 \times [0, \infty)$. *Then the isoperimetric difference of* γ_t*:*

$$L^2(t) - 4\pi A(t) \searrow 0$$

exponentially as $t \to \infty$. *In particular, by the Bonnesen's inequality,* γ_t *converges to a round circle with radius* $\sqrt{\frac{A(0)}{\pi}}$.

Proof Compute

$$\frac{d}{dt}\left(L^2(t) - 4\pi A(t)\right)$$
$$= -4\pi \int_0^{2\pi} \frac{1}{k^{\alpha+1}(\theta, t)} d\theta + 2L(t) \int_0^{2\pi} \frac{1}{k^{\alpha}(\theta, t)} d\theta \leq 0 \quad (29)$$

due to (23).

As $\alpha \geq 1$, by (24) we also have

$$\frac{L^2 - 2\pi A}{L\pi} \int_0^{2\pi} \frac{1}{k^\alpha(\theta, t)} d\theta \leq \int_0^{2\pi} \frac{1}{k^{\alpha+1}(\theta, t)} d\theta, \quad \alpha \geq 1, \tag{30}$$

which implies

$$\frac{d}{dt}\left(L^2(t) - 4\pi A(t)\right)$$

$$\leq -4\pi \frac{L^2 - 2\pi A}{L\pi} \int_0^{2\pi} \frac{1}{k^\alpha(\theta, t)} d\theta + 2L(t) \int_0^{2\pi} \frac{1}{k^\alpha(\theta, t)} d\theta \tag{31}$$

$$= -\frac{2}{L}\left(L^2 - 4\pi A\right) \int_0^{2\pi} \frac{1}{k^\alpha(\theta, t)} d\theta. \tag{32}$$

By Hölder inequality, we know that

$$L(t) = \int_0^{2\pi} \frac{1}{k(\theta, t)} d\theta \le \left(\int_0^{2\pi} \frac{1}{k^{\alpha}(\theta, t)} d\theta \right)^{\frac{1}{\alpha}} \left(\int_0^{2\pi} d\theta \right)^{\frac{\alpha-1}{\alpha}}, \quad \alpha \ge 1.$$

Hence there exists a constant C independent of time so that

$$\int_0^{2\pi} \frac{1}{k^{\alpha}(\theta, t)} d\theta \ge C \quad \text{for all} \quad t \in [0, \infty)$$

and we have estimate same as (27) and the proof is done. □