The Onset Problem for a Thin Superconducting Loop in a Large Magnetic Field

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Joint work with Peter Sternberg

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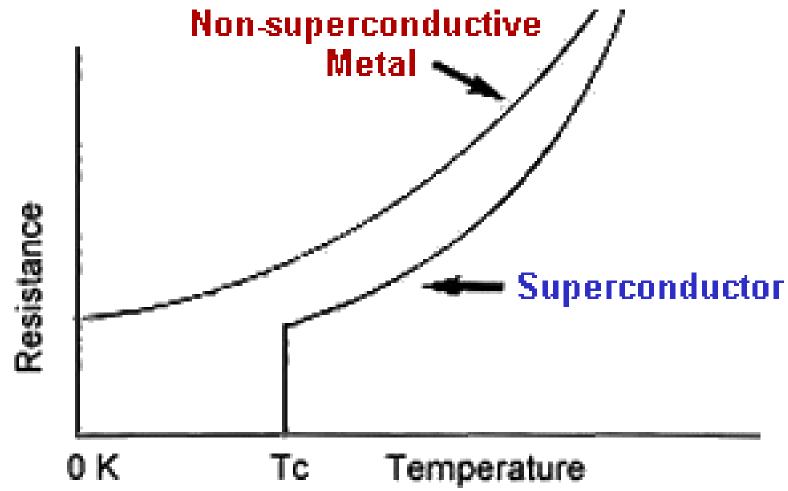
Discovery of Superconductivity

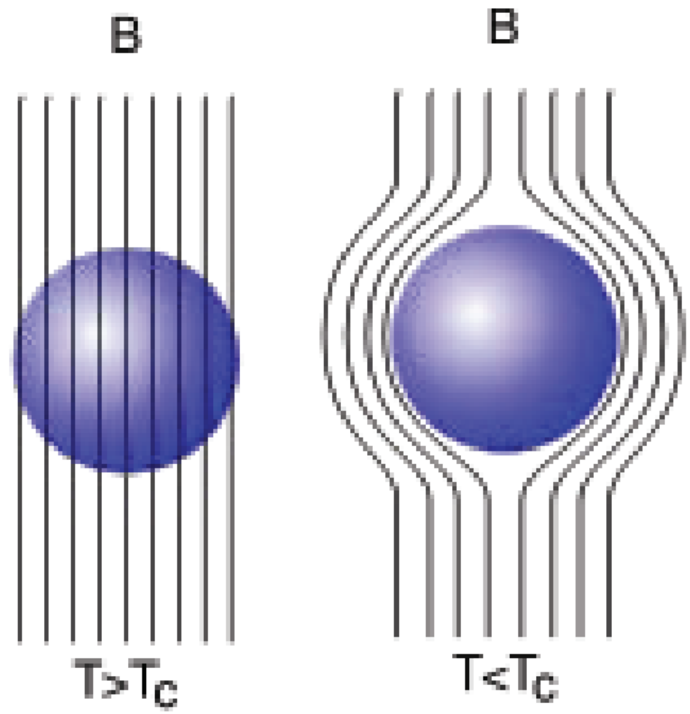
In 1911, superconductivity was first discovered by H. Kamerlingh Onnes in Leiden, just three years after he had first liquefied helium.

Perfect Conductivity



If the temperature is lower than a critical temperature T_c, the electronic resistance completely drops to zero. Once the current is set up, it can stay for very long periods (years).

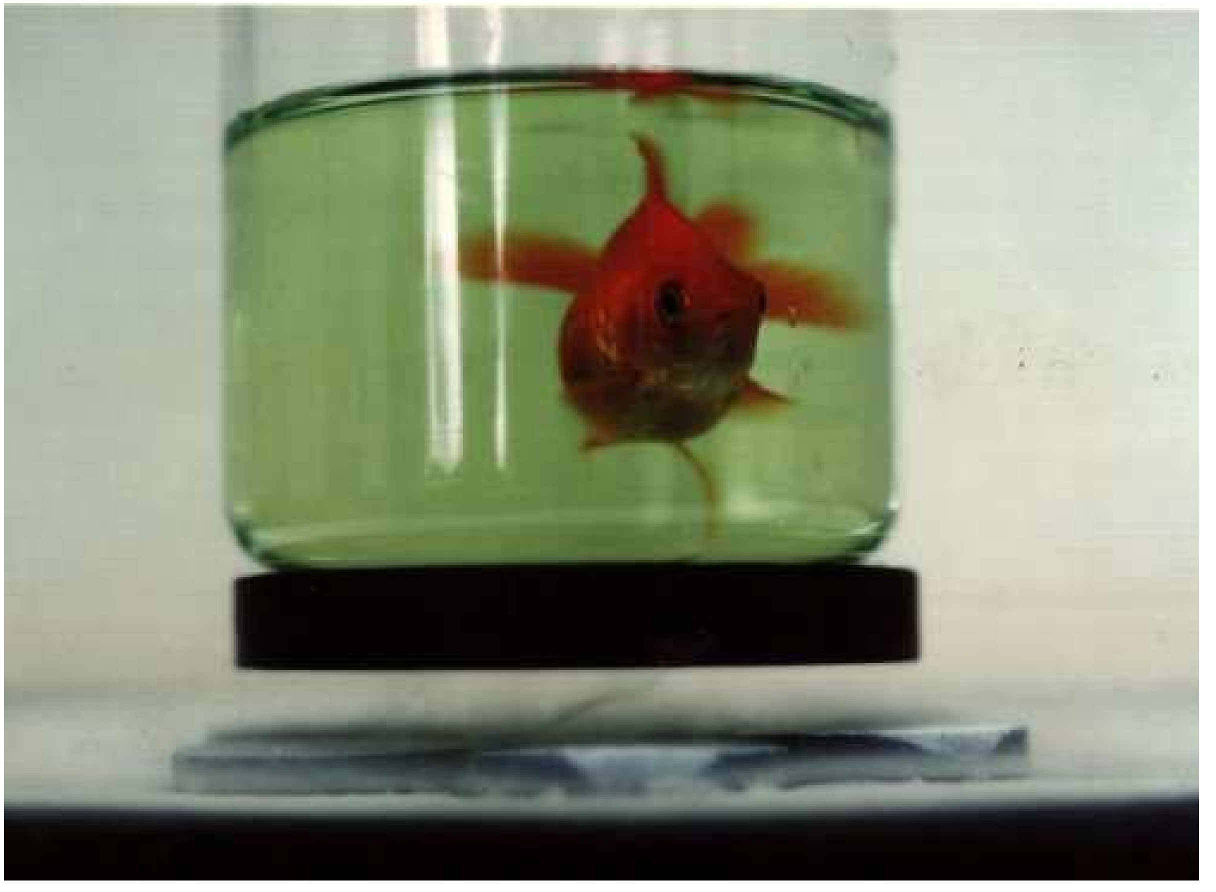
Perfect Diamagnetism



Meissner effect

Perfect diamagnetism was found in 1933 by Meissner and Ochsenfeld. They found a magnetic field is excluded from entering a superconductor.

Levitation: Meissner Effect



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Features of Superconductivity

- Perfect Conductivity

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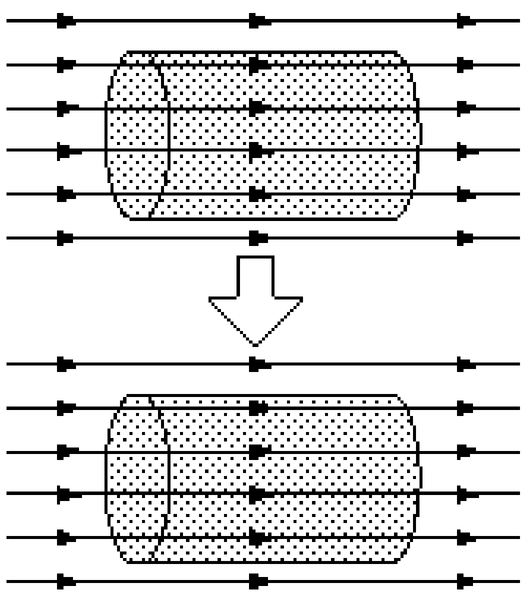

Perfect Conductor

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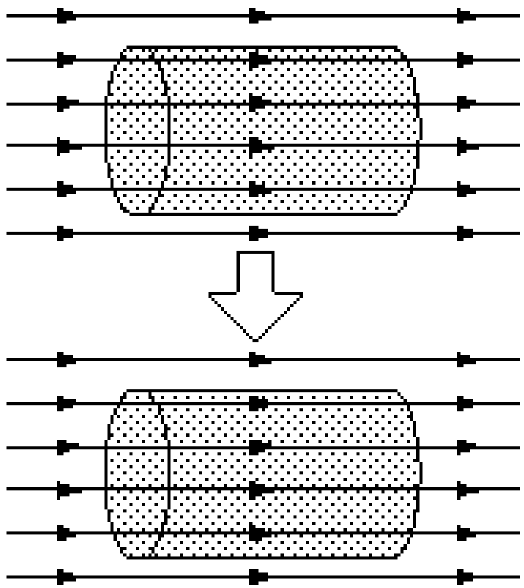

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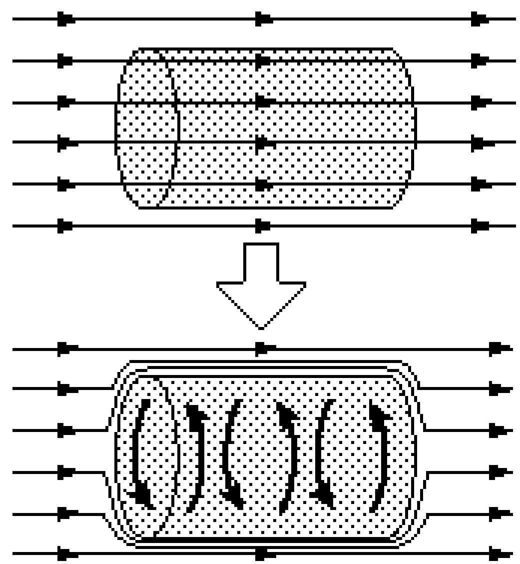
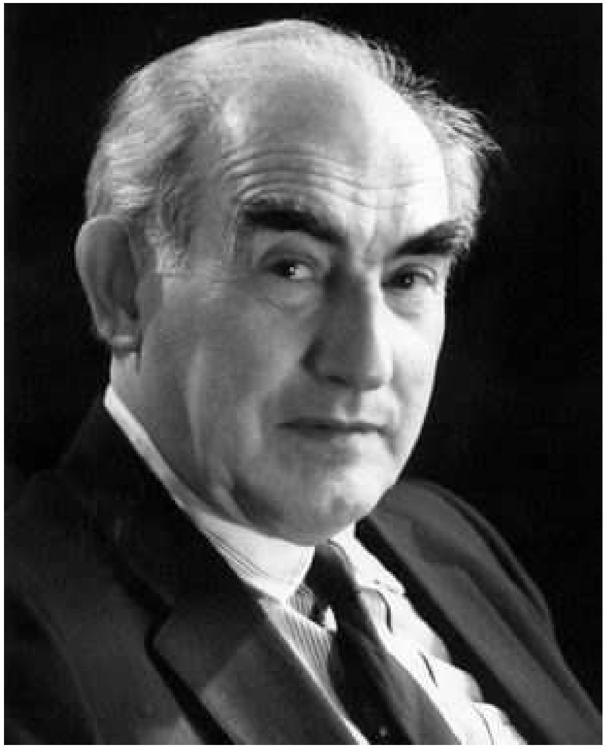


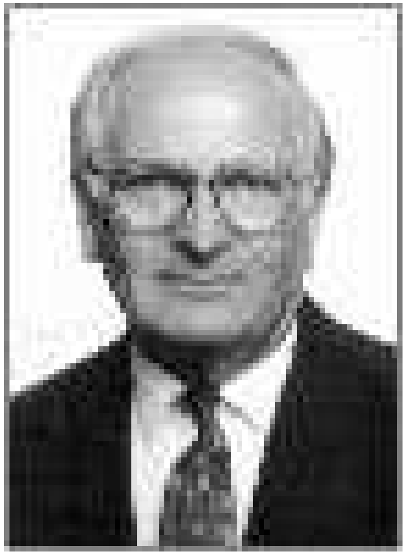
Superconductor

Theories for Superconductivity



In 1950, Ginzburg and Landau propose a phenomenological theory about superconductivity. This theory is based on Landau's general theory of second-order phase transitions.

Theories for Superconductivity

In 1957, Bardeen, Cooper, and Schrieffer propose their microscopic theory about superconductivity, BCS theory. They introduce a idea of ’Cooper pairs’ of electrons.

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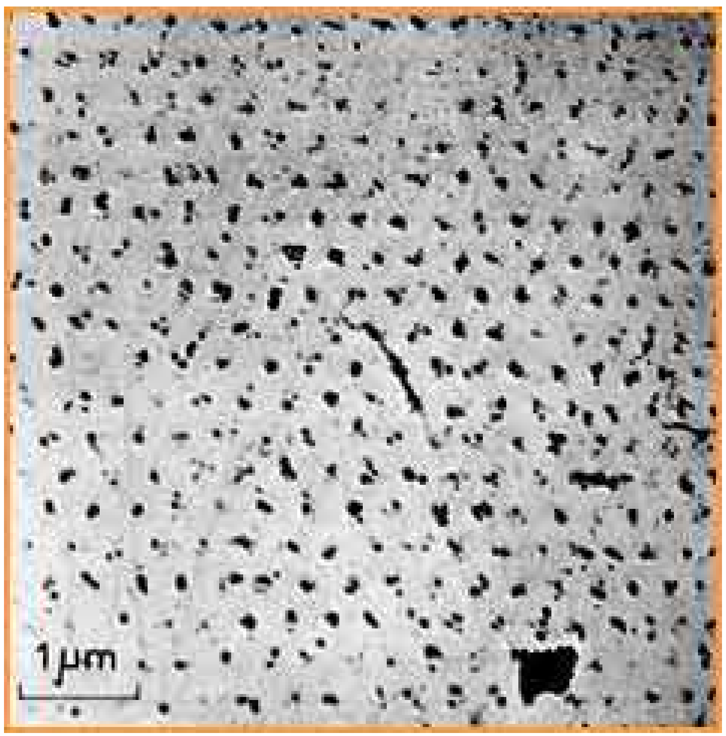
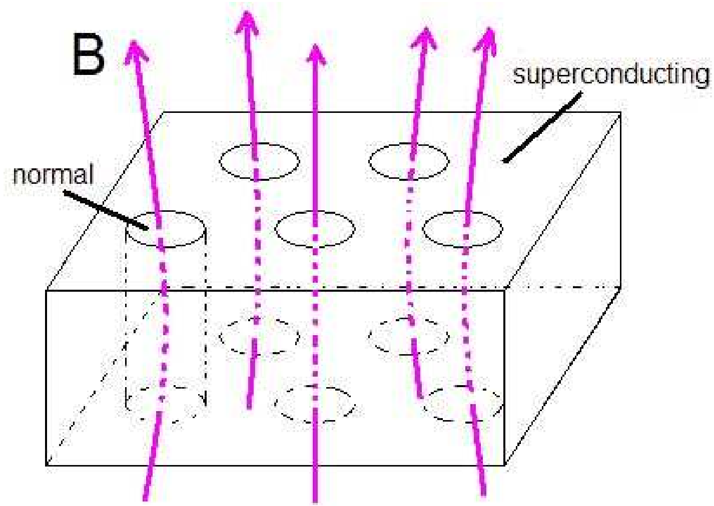
The Glory of the GL Theory

In 1957, Abrikosov predicted the existence of a periodic lattice structure of magnetic flux for Type II superconductor under the framework of the GL theory. Those structures was observed in lab in 1967.





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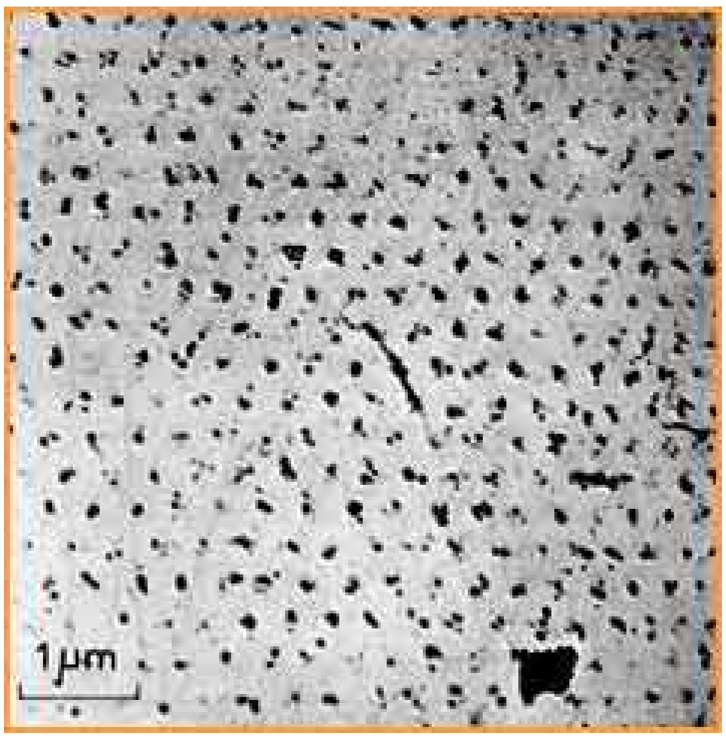
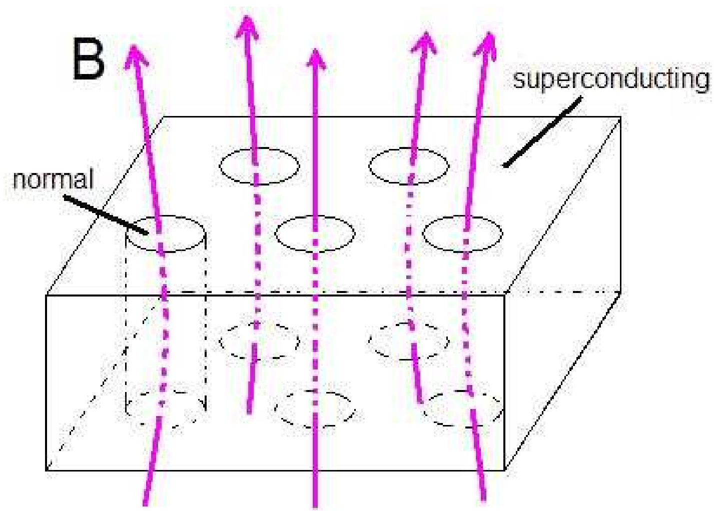
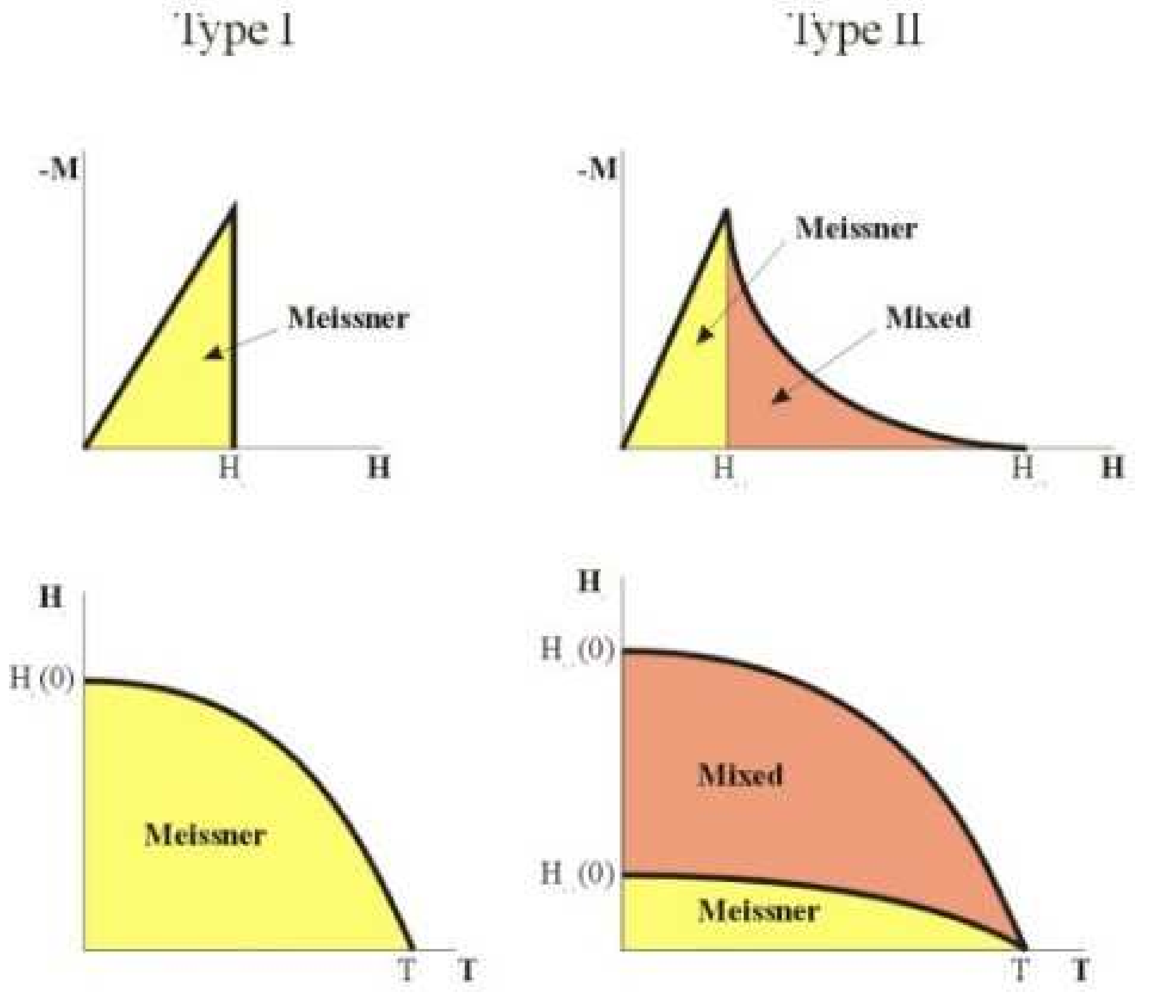




Type I and Type II



BCS and GL Theory

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The BCS well explains the mechanism of the low temperature of superconductivity. However, the BCS theory cannot explains high temperature superconductivity.

The GL theory can even be used to explain some phenomenons for high temperature superconductivity. The GL model is also simpler to handle.

Ginzburg-Landau Model

$$G(\psi, \mathbf{A}) = \int_U \left(|(\nabla - i\mathbf{A})\psi|^2 + \frac{1}{2}\left(|\psi|^2 - \mu^2\right)^2 \right) dx$$

$$+ \int_{\mathbb{R}^3} |\nabla \times \mathbf{A} - \mathbf{H}^e|^2 \, dx.$$

Here $\psi : U \to \mathbb{C}$, with
U: the domain occupied by the sample,
$|\psi|^2$ =density of supercond. charge carriers,
$\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is the induced mag. potential,
T_c : the critical temperature under no applied field,
$\mu^2 \propto T_c - T$, : material constant
$\mathbf{H}^e : \mathbb{R}^3 \to \mathbb{R}^3 =$ given external magnetic field.

Ginzburg-Landau Model

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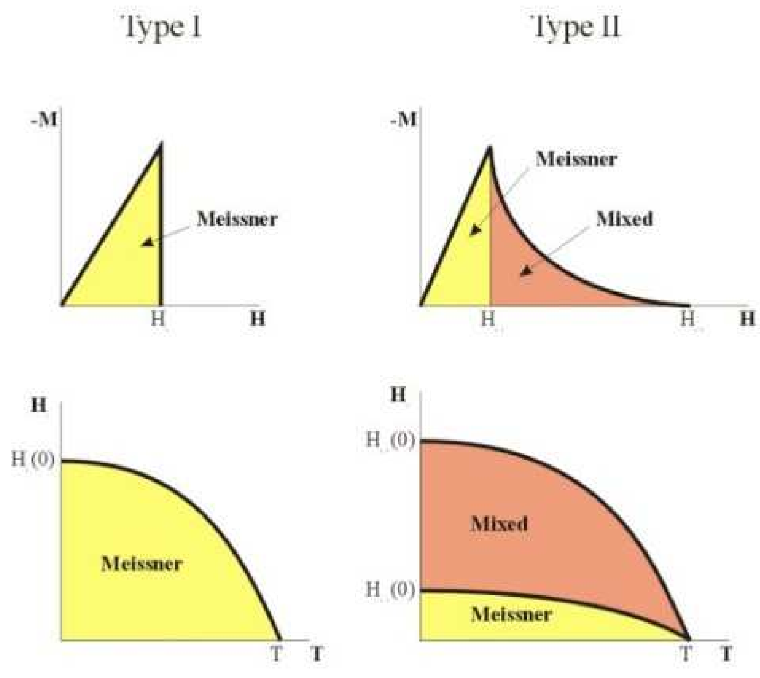
$$+ \int_{\mathbb{R}^3} |\nabla \times \mathbf{A} - \mathbf{H}^e|^2 \, dx.$$

The Basic Thermodynamic Postulate:

The state $(\psi, \mathbf{A})$ of the sample minimizes the Ginzburg-Landau energy.

The Onset of Superconductivity



Normal state : $(\psi, \mathbf{A}) = (0, \mathbf{A}^e)$ where $\nabla \times \mathbf{A}^e = \mathbf{H}^e$.

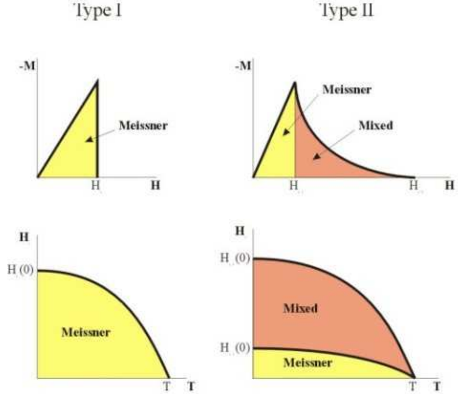
The critical temperature T_c is the phase transition temperature under no applied magnetic field

The Onset of Superconductivity

$$G(\psi, \mathbf{A}) = \int_U \left(|(\nabla - i\mathbf{A})\psi|^2 + \frac{1}{2} \left(|\psi|^2 - \mu^2 \right)^2 \right) dx$$

$$+ \int_{\mathbb{R}^3} |\nabla \times \mathbf{A} - \mathbf{H}^e|^2 \, dx.$$

The phase transition associated with onset of superconductivity is characterized by the value $\mu^2 \quad (\propto (T_c - T))$ at which this normal state loses stability.



The Onset of Superconductivity

Second Variation:

$$\delta^2 G(0, \mathbf{A}_e; \psi, \mathbf{A}) =$$
$$2 \int_U \left(|(\nabla - i\mathbf{A}_e)\psi|^2 - \mu^2 |\psi|^2 \right) dx$$
$$+2 \int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 \, dx.$$

This leads us to consider the following Rayleigh quotient problem:

$$\mu_c^2(\mathbf{H}^e) := \inf_{\psi} \frac{\int_U |(\nabla - i\mathbf{A}^e)\psi|^2 \, dx}{\int_U |\psi|^2 \, dx}.$$

The Onset of Superconductivity

From the second variation, we know

$$\mu_c^2(\mathbf{H}^e) > \mu^2(T) \qquad \text{Normal state is stable}$$
$$\mu_c^2(\mathbf{H}^e) < \mu^2(T) \qquad \text{Normal state is unstable.}$$

Phase Transition curve:

$$\mu_c^2(\mathbf{H^e}) = \mu^2(T) = \alpha(T_c - T) = \alpha\Delta T$$

The Onset Problem

- Superconducting sample in the presence of a large magnetic field $H_e = h\,\mathbf{e}_z$:

When $U = \mathbb{R}^2$, we have

$$\mu_c(h) = h \qquad \text{(Landau)}$$

The Onset Problem

- Superconducting sample in the presence of a large magnetic field $H_e = h\,\mathbf{e}_z$:

When $U = \mathbb{R}^2_+$, we have

$$\mu_c(h) = \lambda_1 h \qquad \text{where } \lambda_1 \approx 0.59$$

The first eigenfunction exponentially decays from the boundary. (Surface superconductivity)

(St. james, De Gennes, Chapman, Lu & Pan)

The Onset Problem

- Superconducting sample in the presence of a large magnetic field $H_e = h\,\mathbf{e}_z$:

When U = smooth bounded domain in R^2

$$\mu_c(h) = \lambda_1 h - \frac{\kappa_{max}}{3I_0} h^{1/2} + o(h^{1/2}) \qquad \text{as } h \to \infty$$

where λ_1 is eigenvalue corresponding to the half-space and I_0 is a universal constant and κ_{max} is the maximal curvature of the boundary. The first eigenfunction is exponentially localized near points of maximal curvature on the boundary.

(P. Bauman, D. Phillips, Q. Tang, A. Bernoff, P. Sternberg, K. Lu, X. Pan, M. del Pino, P. Felmer, B. Helffer, A. Morame)

The Onset Problem

- For $U \subset \mathbb{R}^3$ or in-homogenous magnetic fields:

X. Pan, B. Helffer and A. Morame

Thin Superconducting Samples

J. Rubinstein and M. Schatzman (2001)

They consider the Ginzburg-Landau functional on a ε-neighborhood of a planner embedded graph M. They prove that its minimizers converge in a suitable sense to the minimizers of a simpler functional on the planar graph M.

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- Thin superconducting sample with constriction:

- J. Rubinstein, M. Schatzman,P. Sternberg
- J. Rubinstein, P. Sternberg, G. Wolansky

Question

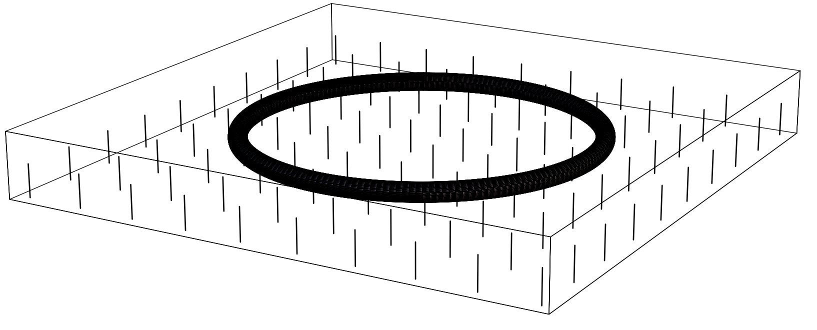
Here we ask :

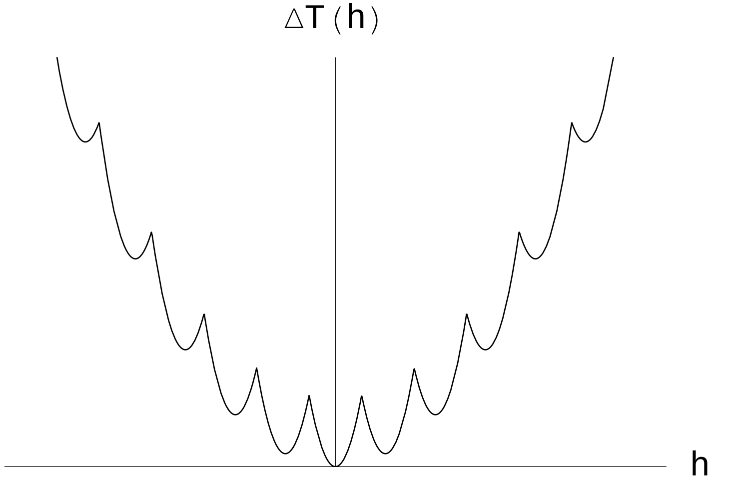
- Can one (rigorously) derive a model of an onset problem for a thin superconducting loop in a presence of large magnetic field starting from three-dimensional Ginzburg-Landau model?

This case was first treated by Richardson and Rubinstein using formal asymptotic expansion.

Little-Parks Experiment

In 1961, Little and Parks observed that the phase transition temperature in thin ring is essentially a periodic function of the axial magnetic flux through the ring with a parabolic background.

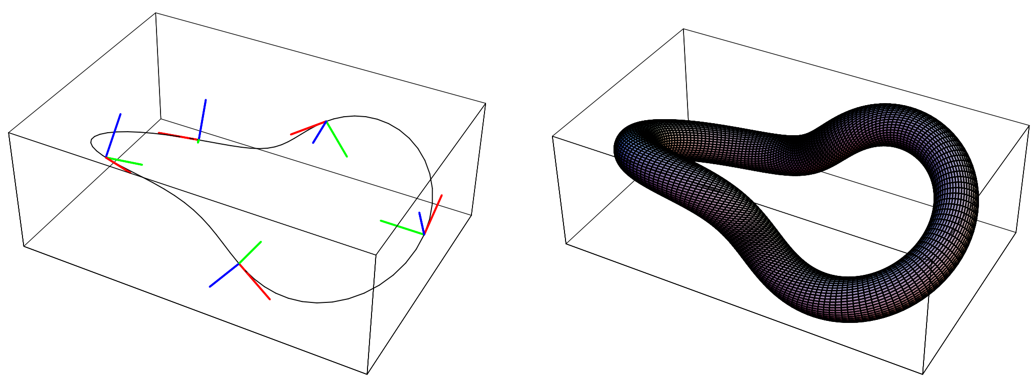




The Model of our problem

- Two Assumptions in our study :

1. The sample domain U is a sequence of domains $\{U_\varepsilon\}$ consisting of ε-neighborhoods of a limiting simple closed curve.



The Model of our problem

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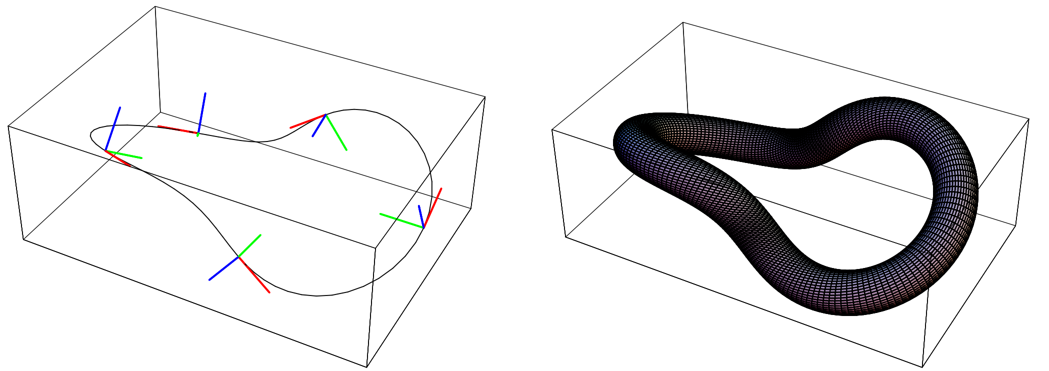
1. The sample domain U is a sequence of domains $\{U_\varepsilon\}$ consisting of ε-neighborhoods of a limiting simple closed curve.

2. The given applied field $\mathbf{H}_\varepsilon$ take the form

$$\mathbf{H}^e_\varepsilon = \frac{\mathbf{H}^e}{\varepsilon}$$

where $\mathbf{H}^e$ is a given smooth magnetic field independent of ε.

Description of U_ε



Let $\mathbf{r} : [0, L] \to \mathbb{R}^3$ be a simple, closed C^2 curve. The triple $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ forms a Frenet Frame for the curve.

The ε-neighborhood U_ε is the image of the cylinder

$$\Omega = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : 0 \leq y_1 \leq L,\ y_2^2 + y_3^2 < 1\}$$

under the mapping

$$T_\varepsilon(y) = \mathbf{r}(y_1) + \varepsilon y_2\, \mathbf{n}(y_1) + \varepsilon y_3\, \mathbf{b}(y_1)\,.$$

Rayleigh quotient problem

This leads us to consider the following Rayleigh quotient problem:

$$\inf_{\psi} \frac{\int_{U_\varepsilon} \left|(\nabla - i\frac{\mathbf{A}^e}{\varepsilon})\psi\right|^2 \, dx}{\int_{U_\varepsilon} |\psi|^2 \, dx}.$$

Main Idea

Set

$$E_\varepsilon(\psi) := \frac{\int_{U_\varepsilon} \left|(\nabla - i\frac{\mathbf{A}^e}{\varepsilon})\psi\right|^2 \, dx}{\int_{U_\varepsilon} |\psi|^2 \, dx}.$$

Main Idea: Identify a limiting energy such that if the minimizers ψ_ε of E_ε converge to some limit ψ_0 defined on the limiting curve, then ψ_0 minimizes this limiting energy.

Main difficulty

$$E_\varepsilon(\psi) := \frac{\int_{U_\varepsilon} \left|(\nabla - i\frac{\mathbf{A}^e}{\varepsilon})\psi\right|^2 dx}{\int_{U_\varepsilon} |\psi|^2 dx}.$$

Main difficulty: The phase of the minimizer ψ_ε oscillates rapidly as $\varepsilon \to 0$. Without shifting the phase, we can't attain compactness of minimizers ψ_ε of E_ε.

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We shift the phase function by special phase functions ϕ_ε.

$$\frac{\mathbf{A}_\varepsilon}{\varepsilon} - \nabla\phi_\varepsilon \approx \mathcal{O}(1) \qquad \text{in } U_\varepsilon \,.$$

Define an equivalent functional

This lead us to find an equivalent energy functional to E_ε.

$$F_\varepsilon(\psi) := E_\varepsilon(\psi e^{i\phi_\varepsilon}) \;=\; \frac{\int_{U_\varepsilon} \left|\left(\nabla - i(\frac{\mathbf{A}^e}{\varepsilon} - \nabla\phi_\varepsilon)\right)\psi\right|^2 dx}{\int_{U_\varepsilon} |\psi|^2 \, dx}$$

Find ϕ_ε:

1. We decompose $\mathbf{A}^e$ into components A_1^e, A_2^e and A_3^e lying along the tangent, normal and bi-normal to the limiting curve $\mathbf{r}$.
2. We set the number k_ε be the closest integer to the number $\left(\frac{1}{L}\int_0^L \frac{A_1^e(y_1,0,0)}{\varepsilon}\, dy_1\right)$.

Special phase function ϕ_ε

Define ϕ_ε by

$$\phi_\varepsilon(y_1,y_2,y_3) := \int_0^{y_1}\left(\frac{A_1(t,0,0)}{\varepsilon}-\beta_\varepsilon\right)dt$$
$$+\frac{1}{\varepsilon}\Bigg(y_2A_2(y_1,0,0)+y_3A_3(y_1,0,0)$$
$$+\frac{y_2^2}{2}\partial_{y_2}A_2(y_1,0,0)+\frac{y_3^2}{2}\partial_{y_3}A_3(y_1,0,0)$$
$$+\frac{1}{2}y_2y_3\partial_{y_3}A_2(y_1,0,0)+\frac{1}{2}y_2y_3\partial_{y_2}A_3(y_1,0,0)\Bigg)$$

where the effective magnetic flux

$$\beta_\varepsilon = \left(\frac{1}{L}\int_0^L \frac{A_1(t,0,0)}{\varepsilon}dt\right)-\frac{2\pi}{L}k_\varepsilon.$$

The equivalent functional F_ε

$$F_\varepsilon(\psi) \quad = \quad \frac{\int_\Omega \left| \frac{1}{\eta_\varepsilon}\left(\partial_{y_1}\psi + \tau y_3 \partial_{y_2}\psi - \tau y_2 \partial_{y_3}\psi\right)\right.}{\int_\Omega |\psi|^2 \eta_\varepsilon \, dy}$$

$$\frac{\left. - i\left(\frac{\beta_\varepsilon}{\eta_\varepsilon} - y_2 H_3^e + y_3 H_2^e\right)\psi - \frac{1}{\varepsilon} R^\varepsilon \psi \right|^2 \eta_\varepsilon \, dy}{\int_\Omega |\psi|^2 \eta_\varepsilon \, dy}$$

$$+ \frac{\int_\Omega \left| \frac{1}{\varepsilon}\partial_{y_2}\psi + i\left(\frac{1}{2} y_3\right) H_1^e \psi - \frac{i}{\varepsilon} R_2^\varepsilon \psi \right|^2 \eta_\varepsilon \, dy}{\int_\Omega |\psi|^2 \eta_\varepsilon \, dy}$$

$$+ \frac{\int_\Omega \left| \frac{1}{\varepsilon}\partial_{y_3}\psi - i\left(\frac{1}{2} y_2\right) H_1^e \psi - \frac{i}{\varepsilon} R_3^\varepsilon \psi \right|^2 \eta_\varepsilon \, dy}{\int_\Omega |\psi|^2 \eta_\varepsilon \, dy}$$

where $\eta_\varepsilon = 1 - \varepsilon \kappa y_2$ and κ is the curvature of the curve $\mathbf{r}$.

The limiting energy G_β

We guess limiting energy should takes the following form:

$$G_\beta(\psi) := \frac{\int_0^L |\left(\frac{d}{dy_1} - i\beta\right)\psi|^2 + W(y_1)|\psi|^2 dy_1}{\int_0^L |\psi|^2 dy_1}.$$

Here

$$W(y_1) := \frac{1}{8}(H_1^e)^2 + \frac{1}{4}(H_2^e)^2 + \frac{1}{4}(H_3^e)^2.$$

Theorem 1

We obtain a compactness result

Theorem 1 Let ψ_ε is the minimizer of F_ε in $H^1(\Omega)$. There exists a subsequence $\{\psi_{\varepsilon_j}\}$ and $\psi_0 \in H^1(\Omega)$ and $\beta_0 \in [-\frac{\pi}{L}, \frac{\pi}{L}]$ such that

$$\begin{aligned} \psi_{\varepsilon_j} &\rightharpoonup \psi_0 && \text{weakly in } H^1(\Omega), \\ \psi_{\varepsilon_j} &\to \psi_0 && \text{strongly in } L^q(\Omega), 1 \le q < 6 \\ \beta_{\varepsilon_j} &\to \beta_0 \end{aligned}$$

Note that ψ_0 is a function of y_1 only.

Theorem 2

Involving techniques of dimension reduction and Γ-convergence. We obtain

Theorem 2 The limiting function $\psi_0 \in H^1((0,L))$ minimizes G_{β_0}.

This result is followed by the claim

$$G_{\beta_0}(\psi_0) \leq \liminf_{\varepsilon_j \to 0} F_{\varepsilon_j}(\psi_{\varepsilon_j}).$$

All terms in F_{ε_j} will converge to their 1d analogs.

Theorem 3

Comparing the minimum of F_ε and the minimum of G_{β_ε}. This gives us

Theorem 3 Let λ_ε be the minimum of F_ε and let σ_ε be the minimum of G_{β_ε}. Then

$$(\lambda_\varepsilon - \sigma_\varepsilon) = \mathcal{O}(\varepsilon)\,.$$

Theorem 4

Asymptotic relationship between the first eigenspaces of functionals F_ε and G_{β_ε}:

Theorem 4 Let $\varepsilon_j \to 0$ be any sequence such that

$$-\frac{\pi}{L} < \liminf_{j\to\infty} \beta_{\varepsilon_j} \le \limsup_{j\to\infty} \beta_{\varepsilon_j} < \frac{\pi}{L}\,.$$

and let ψ^{ε_j} be a minimizer of F_{ε_j} in $H^1(\Omega)$ with $\|\psi^{\varepsilon_j}\|_{L^2(\Omega)} = 1$. Then there exists a sequence $\{\psi_0^{\varepsilon_j}\}$ minimizing $\{G_{\beta_{\varepsilon_j}}\}$ in $H^1_{\rm per}((0,L))$ with $\|\psi_0^{\varepsilon_j}\|_{L^2(\Omega)} = 1$ such that

$$\psi^{\varepsilon_j} - \psi_0^{\varepsilon_j} \;\to\; 0 \quad \text{strongly in } H^1(\Omega)\,.$$

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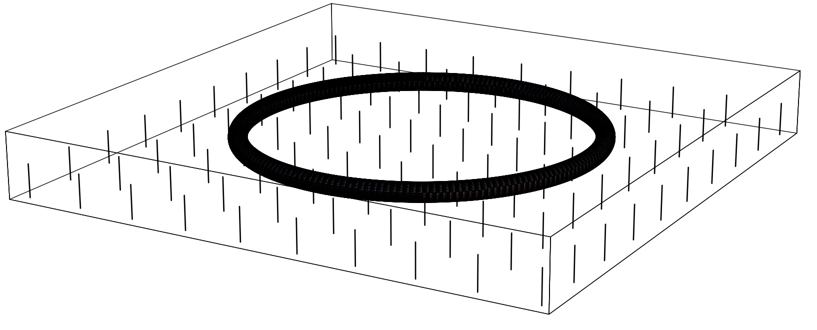

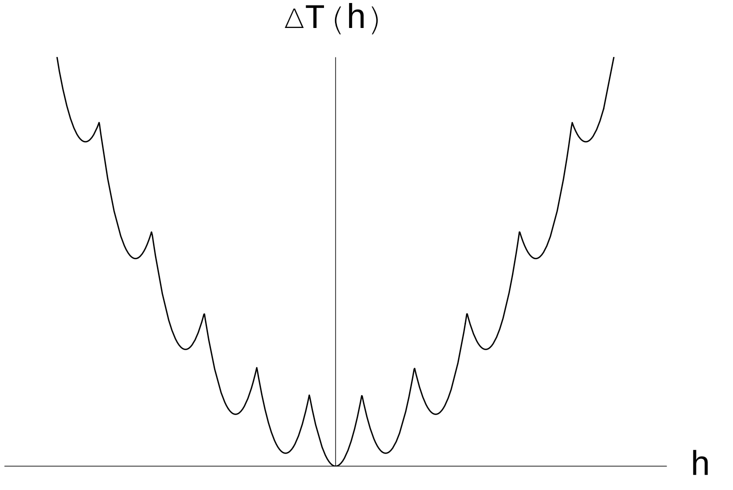

N-S Transition Curve

$$\mu_c^2(h\mathbf{e}_z)$$

$$\sim \inf_{\psi} G_{\beta_0}(\psi)$$

$$= \inf_{\psi} \frac{\int_0^L |\left(\frac{d}{dy_1} - i\beta_0(h)\right)\psi|^2 + \frac{h^2}{4}|\psi|^2 dy_1}{\int_0^L |\psi|^2 dy_1}.$$

$$= \beta_0(h) + \frac{1}{4}h^2$$

N-S Transition Curve

The result of the Little-Parks experiment can be found through the relation

$$\mu_c^2(h\,\mathbf{e}_z) = \mu^2(T) = \alpha(T_c - T) = \alpha\Delta T\,,$$

we obtain

$$\beta_0(h) + \frac{1}{4}\,h^2 = \mu_c^2(h\,\mathbf{e}_z) = \alpha\Delta T\,.$$

where

$$\mu_c^2(\mathbf{H}^e) := \inf_{\psi} \frac{\int_U |(\nabla - i\mathbf{A}^e)\psi|^2\,dx}{\int_U |\psi|^2\,dx}.$$

Thank You!