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Convex Relaxations and Mixed-Integer Quadratic Programming Reformulations for Cardinality Constrained Quadratic Programs

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Outline

- Problem & background
- Research motivation
- Convex relaxations via Lagrangian decomposition
- Second-order relaxation and MIQP reformulation
- Preliminary computational results
- Discussions and further study

Cardinality constrained quadratic program

- Optimization model:

$$
\begin{aligned}
(\mathrm{P}) \qquad & \min \; q(x) = \frac{1}{2}x^T Qx + c^T x && \text{(convex quadratic function)}\\
& \text{s.t. } Ax \le b, && \text{(linear constraints)}\\
& \quad |\mathrm{supp}(x)| \le K, && \text{(cardinality)}\\
& \quad x_i \ge \alpha_i, \; i \in \mathrm{supp}(x), && \text{(minimum positive value)}\\
& \quad 0 \le x \le u,
\end{aligned}
$$

where $\mathrm{supp}(x) = \{i \mid x_i \neq 0\}$, Q is a positive semidefinite matrix, $c \in \mathbb{R}^n$, $A \in \Re^{m\times n}$, $b \in \Re^m$, $1 \le K \le n$ is an integer, $0 < \alpha_i < u_i$.

- Difficulty: testing the feasibility of (P) is already NP-complete when A has three rows (Bienstock (1996)).

Portfolio selection with cardinality constraint

- Markowitz's classical mean-variance model:

$$\begin{aligned} \min\ & x^T \Sigma x \\ \text{s.t. } & e^T x = 1, \\ & \mu^T x \geq \rho, \end{aligned}$$

  where μ is the expected return $\mu = E(r)$ for the random return vector r, Σ is the covariance matrix $\Sigma = E[(r - \bar{r})(r - \bar{r})^T]$ and ρ is the target return.
- Cardinality constraint: the number of assets in the optimal portfolio should be limited:

$$|\text{supp}(x)| \leq K,$$

  The need to account for this limit is due to the transaction cost and managerial concerns.

Portfolio selection and index tracking

- Portfolio section with cardinality and buy-in threshold constraints:

$$
\begin{aligned}
(\mathrm{P}) \qquad & \min \; q(x) = x^T \Sigma x - r_a \, \mu^T x \quad \text{(risk or utility)} \\
& \text{s.t. } Ax \le b, \quad \text{(return, budget, sector, regulations)} \\
& \quad |\mathrm{supp}(x)| \le K, \quad \text{(cardinality)} \\
& \quad x_i \ge \alpha_i, \; i \in \mathrm{supp}(x), \quad \text{(buy-in threshold)} \\
& \quad 0 \le x \le u, \quad \text{(bounds on positions, no short-selling)}.
\end{aligned}
$$

- Index tracking:

$$\text{tracking error} = (x - x_0)^T \Sigma (x - x_0),$$

where x is the trading vector with small number of nonzero variables and x_0 is the weight vector of the benchmark index (S&P 500, FTSE 100, N225).

Literature review

- Jacob (1974, J. Finance). Limited-diversified portfolio selection model for small investors.
- Blog et al. (1984, Manage. Sci.), Portfolio selection of small portfolio, dynamic heuristic method.
- Bonami and Lejeune (2009, OR), exact methods for portfolio optimization problems under stochastic and discrete constraints including cardinality and minimum buy-in threshold.
- Branch-and-bound and branch-and-cut methods based on continuous relaxations. Bienstock (1996, MP), Bertsimas and Shioda (2009, COA), Li, Sun and Wang (2006, MF), Shawa et al. (2008, OMS).
- Various heuristic methods. Chang et al. (2001, EJOR), Maringer and Kellerer (2003, OR Spectrum), Mitra et al. (2007, JAM).

Standard mixed-integer QP reformulation

- Introducing 0-1 variables $y_i \in \{0,1\}$, (P) can be reformulated as

$$
\begin{aligned}
(\mathrm{MIQP}) \qquad \min\ & q(x) = \frac{1}{2}x^T Qx + c^T x \\
\text{s.t. } & Ax \le b, \\
& e^T y \le K,\ y \in \{0,1\}^n, \\
& x_i^2 - (\alpha_i + u_i)x_i + \alpha_i u_i y_i \le 0,\ i = 1,\dots,n, \\
& 0 \le x_i \le u_i y_i,\ i = 1,\dots,n,
\end{aligned}
$$

- $y \in \{0,1\} \Rightarrow y \in [0,1]$. The continuous relaxation (QP).

Research Motivation

Can we do better than the standard MIQP reformulation?

- Does there exist tighter convex relaxations of (P) than the continuous relaxation of (MIQP_0)? $\Rightarrow$ SDP and SCOP.
- Does there exist a more efficient reformulation of mixed integer QP for (P) than (MIQP_0)?

- Relaxing: $X = xx^T \Rightarrow X \succeq xx^T$, $Y = yy^T \Rightarrow Y \succeq yy^T$. SDP relaxation of (MIQP):

$$
\begin{aligned}
(\mathrm{SDP}_0) \qquad & \min \ \frac{1}{2} Q \bullet X + c^T x \\
& \text{s.t. } Ax \le b, \ 0 \le x_i \le u_i y_i, \ i = 1, \ldots, n, \\
& \quad X_{ii} - (\alpha_i + u_i) x_i + \alpha_i u_i y_i \le 0, \ i = 1, \ldots, n, \\
& \quad e^T y \le K, \ Y_{ii} = y_i, \ i = 1, \ldots, n, \\
& \quad \begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix} \succeq 0, \ \begin{pmatrix} Y & y \\ y^T & 1 \end{pmatrix} \succeq 0.
\end{aligned}
$$

- $v(SDP_0) = v(QP)$!. (SDP_0) is the conic dual of the conventional Lagrangian relaxation (dualizing all constraints).
- Stronger Lagrangian dual formulations: Lagrangian decomposition scheme (Guignard and Kim (1987), Michelon and Maculan (1991),Shawa et al. (2008)).

SDP Relaxations: Lagrangian decomposition

- Main ideas:
  - Decompose Q as $Q = (Q - \mathrm{diag}(d)) + \mathrm{diag}(d)$, where $Q - \mathrm{diag}(d) \succeq 0$;
  - Construct a convex relaxation of (P) by Lagrangian decomposition technique via copying constraints;
  - Reduce the Lagrangian dual to an SDP formulation.
- The objective function decomposition:

$$\frac{1}{2}x^T \mathrm{diag}(d) x + c^T x + \frac{1}{2}x^T (Q - \mathrm{diag}(d)) x.$$

Lagrangian relaxation via diagonal decomposition and copying constraints

$$
\begin{aligned}
& & \min\ & \frac{1}{2}x^T \mathrm{diag}(d) x + c^T x + \frac{1}{2} z^T (Q - \mathrm{diag}(d)) z \\
\pi: & & \text{s.t.}\ & Az \le b,\ 0 \le z \le u, \\
\lambda: & & & x = z, \qquad \text{(link constraint)} \\
& & & |\mathrm{supp}(x)| \le K,\ 0 \le x \le u, \\
& & & x_i \ge \alpha_i,\ i \in \mathrm{supp}(x).
\end{aligned}
$$

- Lagrangian relaxation:

$$d(\pi,\lambda) = -\pi^T b + d_1(\pi,\lambda) + d_2(\pi,\lambda),$$

where

$$\begin{aligned} d_1(\pi,\lambda) &= \min\ \frac{1}{2}x^T \mathrm{diag}(d)x + (c^T + \pi^T A - \lambda^T)x \\ & \quad \text{s.t. } |\mathrm{supp}(x)| \le K,\ 0 \le x \le u, \\ & \quad\quad x_i \ge \alpha_i,\ i \in \mathrm{supp}(x), \\ d_2(\pi,\lambda) &= \min\ \frac{1}{2}z^T (Q - \mathrm{diag}(d))z + \lambda^T z, \\ & \quad \text{s.t. } Az \le b,\ 0 \le z \le u. \end{aligned}$$

- Lagrangian dual:

$$(D_d) \quad \max\{-\pi^T b + d_1(\pi,\lambda) + d_2(\pi,\lambda) \mid (\pi,\lambda) \in \Re^m_+ \times \Re^n\}.$$

- Dual bound via best diagonal decomposition:

$$(D) \quad \max_{Q-\mathrm{diag}(d)\succeq 0} v(D(d)).$$

- Denote the sum of the K largest elements of $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ by $S_K(x) = \sum_{k=1}^{K} x_{i_k}$, Then

$$d_1(\pi, \lambda) = \max\{-t \mid S_K(-q) \leq t\}.$$

- $S_K(p) \leq t$ is SDP representable (Nemirovski (2001)):

$$\begin{array}{ll} (a) & t - Ks - e^{\mathrm{T}} z \geq 0, \\ (b) & z \geq 0, \\ (c) & z - p + se \geq 0. \end{array}$$

- Problem (D) is equivalent to the following SDP problem (DSDP):

$$
\begin{aligned}
\max\ & -\pi^T b - t + \gamma \\
\text{s.t. } & \begin{pmatrix} d_i + 2\mu_i & \tilde{c}_i - \mu_i(\alpha_i + u_i) \\ \tilde{c}_i - \mu_i(\alpha_i + u_i) & -2\tau_i + 2\mu_i\alpha_i u_i \end{pmatrix} \succeq 0,\ i = 1, \dots, n, \\
& \begin{pmatrix} Q - \mathrm{diag}(d) & \lambda + A^T\eta - \zeta + \xi \\ \lambda^T + \eta^T A - \zeta^T + \xi^T & -2\eta^T b - 2\xi^T u - 2\gamma \end{pmatrix} \succeq 0, \\
& \tau - \beta \geq 0,\ t - Ks - e^T z \geq 0,\ z + \beta + se \geq 0, \\
& (t, s, z, \mu, \tau, -\beta) \in \Re \times \Re \times \Re^n \times \Re^n_+ \times \Re^n \times \Re^n_+, \\
& (\gamma, \eta, \xi, \zeta) \in \Re \times \Re^m_+ \times \Re^n_+ \times \Re^n_+,\ (\pi, \lambda) \in \Re^m_+ \times \Re^n.
\end{aligned}
$$

- If $Q \succeq 0$, then

$$
v(\mathrm{QP}) \leq v(\mathrm{D_d}) \leq v(\mathrm{D}) = v(\mathrm{DSDP}).
$$

The conic dual of (DSDP) is

$$
(\mathrm{SDP}_1) \qquad \begin{aligned}
\min\ & \frac{1}{2} Q \bullet X + c^{\mathrm{T}} x \\
\text{s.t. } & Ax \le b, \\
& \phi_i - (\alpha_i + u_i) x_i + \alpha_i u_i y_i \le 0,\ i = 1, \ldots, n, \\
& 0 \le x \le u, \\
& y \in [0,1]^n,\ e^{\mathrm{T}} y \le K, \\
& X_{ii} = \phi_i,\ i = 1, \ldots, n, \\
& \begin{pmatrix} X & x \\ x^{\mathrm{T}} & 1 \end{pmatrix} \succeq 0,\ \begin{pmatrix} \phi_i & x_i \\ x_i & yi \end{pmatrix} \succeq 0,\ i = 1, \ldots, n, \\
& X \in \mathcal{S}^n,\ x, y, \phi \in \Re^n.
\end{aligned}
$$

where $X \in \mathcal{S}^n$, $x, y, \phi \in \Re^n$. (Straightforward yet tedious!)

- $(\mathrm{SDP}_1) \Leftarrow$ (relaxed from) a new reformulation of (MIQP_0):

$$
\begin{aligned}
(\mathrm{MIQP}_1) \qquad & \min \ \frac{1}{2} Q \bullet X + c^{\mathrm{T}} x \\
& \text{s.t. } Ax \le b, \\
& \qquad \phi_i - (\alpha_i + u_i) x_i + \alpha_i u_i y_i \le 0, \ i = 1, \dots, n, \\
& \qquad 0 \le x \le u, \\
& \qquad y \in \{0,1\}^n, \ e^{\mathrm{T}} y \le K, \\
& \qquad \phi_i = x_i^2, \ i = 1, \dots, n, \\
& \qquad X = xx^{\mathrm{T}}, \ \phi_i y_i = x_i^2, \ \phi_i \ge 0, \ i = 1, \dots, n.
\end{aligned}
$$

- $x_i^2 \Rightarrow X_{ii}$, $X = xx^T \Rightarrow X \succeq xx^T$, $y_i \in \{0,1\} \Rightarrow y_i \in [0,1]$, $\phi_i y_i = x_i^2 \Rightarrow \phi_i y_i \ge x_i^2$.

$$
\phi_i y_i \ge x_i^2, \ \phi_i \ge 0, \ y_i \ge 0 \Leftrightarrow \begin{pmatrix} \phi_i & x_i \\ x_i & y_i \end{pmatrix} \succeq 0, \ i = 1, \dots, n.
$$

Second-order cone relaxation

- Computational difficulties arise in solving SDP problem with large-size matrix variables. SOCP is a reasonable compromise between SDP and LP relaxation (lift-and-project). Powerful SOCP solvers are available (CPLEX, MOSEK).
- $(\mathrm{D_d})$ can be expressed as:

$$
\begin{aligned}
\max \quad & -\pi^{\mathrm{T}} b - t + \gamma \\
\text{s.t.} \ \Upsilon_i := & \begin{pmatrix} d_i + 2\mu_i & \tilde{c}_i - \mu_i(\alpha_i + u_i) \\ \tilde{c}_i - \mu_i(\alpha_i + u_i) & -2\tau_i + 2\mu_i\alpha_i u_i \end{pmatrix} \succeq 0, \ i = 1, \ldots, n, \\
& \Phi := \begin{pmatrix} Q - \mathrm{diag}(d) & \lambda + A^{\mathrm{T}}\eta - \zeta + \xi \\ \lambda^{\mathrm{T}} + \eta^{\mathrm{T}} A - \zeta^{\mathrm{T}} + \xi^{\mathrm{T}} & -2\eta^{\mathrm{T}} b - 2\xi^{\mathrm{T}} u - 2\gamma \end{pmatrix} \succeq 0, \\
& \tau - \beta \geq 0, \ t - Ks - e^{\mathrm{T}} z \geq 0, \ z + \beta + se \geq 0, \\
& (t, s, z, \mu, \tau, -\beta) \in \Re \times \Re \times \Re^n_+ \times \Re^n_+ \times \Re^n \times \Re^n_+, \\
& (\gamma, \eta, \xi, \zeta) \in \Re \times \Re^m_+ \times \Re^n_+ \times \Re^n_+, \ (\pi, \lambda) \in \Re^m_+ \times \Re^n,
\end{aligned}
$$

where $\tilde{c}_i = c_i + a_i^{\mathrm{T}} \pi - \lambda_i$.

- The first LMI is equivalent to

$$\left\| \begin{array}{c} c_i + a_i^T \pi - \lambda_i c_i - \mu_i(u_i + \alpha_i) \\ \frac{d_i + 2\mu_i + 2\tau_i - 2\mu_i \alpha_i u_i}{2} \end{array} \right\|_2 \leq \frac{d_i + 2\mu_i - 2\tau_i + 2\mu_i \alpha_i u_i}{2}.$$

- The second LMI is equivalent to

$$U_i^T(\lambda + A^T\eta - \zeta + \xi) = 0, i = 1, \ldots, r,$$

$$\left\| \begin{array}{c} \frac{U_{r+1}^T(\lambda + A^T\eta - \zeta + \xi)}{\sqrt{\sigma_{r+1}}} \\ \vdots \\ \frac{U_n^T(\lambda + A^T\eta - \zeta + \xi)}{\sqrt{\sigma_n}} \\ \frac{-2\eta^T b - 2\xi^T u - 2\gamma - 1}{2} \end{array} \right\|_2 \leq \frac{-2\eta^T b - 2\xi^T u - 2\gamma + 1}{2}.$$

Therefore, $(\mathrm{D_d}) \Leftrightarrow (\mathrm{DSDP_d})$:

$$
\begin{aligned}
\max \quad & -\pi^{\mathrm{T}} b - t + \gamma \\
\text{s.t.} \quad & \left\| \begin{pmatrix} c_i + a_i^{\mathrm{T}}\pi - \lambda_i c_i - \mu_i(u_i + \alpha_i) \\ \frac{d_i + 2\mu_i + 2\tau_i - 2\mu_i\alpha_i u_i}{2} \end{pmatrix} \right\|_2 \le \frac{d_i + 2\mu_i - 2\tau_i + 2\mu_i\alpha_i u_i}{2}, \\
& \left\| \begin{pmatrix} \frac{U_{r+1}^{\mathrm{T}}(\lambda + A^{\mathrm{T}}\eta - \zeta + \xi)}{\sqrt{\sigma_{r+1}}} \\ \vdots \\ \frac{U_n^{\mathrm{T}}(\lambda + A^{\mathrm{T}}\eta - \zeta + \xi)}{\sqrt{\sigma_n}} \\ \frac{-2\eta^{\mathrm{T}} b - 2\xi^{\mathrm{T}} u - 2\gamma - 1}{2} \end{pmatrix} \right\|_2 \le \frac{-2\eta^{\mathrm{T}} b - 2\xi^{\mathrm{T}} u - 2\gamma + 1}{2}, \\
& U_i^{\mathrm{T}}(\lambda + A^{\mathrm{T}}\eta - \zeta + \xi) = 0, \ i = 1, \ldots, r, \\
& \tau - \beta \ge 0, \ t - Ks - e^{\mathrm{T}} z \ge 0, \ z + \beta + se \ge 0, \\
& (t, s, z, \mu, \tau, -\beta) \in \Re \times \Re \times \Re_+^n \times \Re_+^n \times \Re^n \times \Re_+^n, \\
& (\gamma, \eta, \xi, \zeta) \in \Re \times \Re_+^m \times \Re_+^n \times \Re_+^n, \ (\pi, \lambda) \in \Re_+^m \times \Re^n.
\end{aligned}
$$

- The conic dual of $(\mathrm{DSDP_d})$ is:

$$
(\mathrm{SOCP_d}) \qquad \begin{aligned}
\min\ & c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}(Q - \mathrm{diag}(d))x + \frac{1}{2}\phi^{\mathrm{T}}d \\
\text{s.t. } & Ax \le b,\ 0 \le x \le u, \\
& e^{\mathrm{T}}y \le K,\ 0 \le y \le 1, \\
& \phi_i - (\alpha_i + u_i)x_i + \alpha_i u_i y_i \le 0,\ i = 1, \dots, n, \\
& \left\| \begin{matrix} x_i \\ \frac{\phi_i - y_i}{2} \end{matrix} \right\|_2 \le \frac{\phi_i + y_i}{2},\ i = 1, \dots, n.
\end{aligned}
$$

- For any fixed $d \in \Re^n$ with $Q - \mathrm{diag}(d) \succeq 0$, it holds that $v(\mathrm{SOCP_d}) = v(\mathrm{DSOCP_d}) = v(\mathrm{DSDP_d}) = v(\mathrm{D_d})$.

- $(\mathrm{SOCP_d})$ suggests the following MIQP problem of (P):

$$(\mathrm{MIQP_d}) \qquad \begin{aligned} \min\ & c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}(Q - \mathrm{diag}(d))x + \frac{1}{2}\phi^{\mathrm{T}}d \\ \text{s.t. } & Ax \le b,\ 0 \le x \le u, \\ & e^{\mathrm{T}}y \le K,\ y \in \{0,1\}^n, \\ & \phi_i - (\alpha_i + u_i)x_i + \alpha_i u_i y_i \le 0,\ i = 1,\dots,n, \\ & x_i^2 \le \phi_i y_i,\ \phi_i \ge 0,\ i = 1,\dots,n. \end{aligned}$$

- The continuous relaxation of $(\mathrm{MIQP_d})$ is exactly $(\mathrm{SOCP_d})$.
- For any $d \in \Re^n_+$, $v(\mathrm{MIQP_d}) = v(\mathrm{MIQP_0}) = v(\mathrm{P})$.
- Since $v(\mathrm{SOCP_d}) \ge v(QP)$ for any $d \ge 0$, $(\mathrm{MIQP_d})$ is a more efficient reformulation for cardinality constrained QP.

Preliminary computational results

- Purpose of the numerical experiment:
  - Comparison of the SDP and SOCP bounds for cardinality constrained QP;
  - Comparison of the effectiveness of the new MIQP reformulation.
- Computnig environment:
  - Convex mixed integer QCP solver in CPLEX 12.1 with Matlab interface is used to solve the MIQP reformulations $(\mathrm{MIQP_d})$ and $(\mathrm{MIQP_0})$ of (P).
  - The SDP problems (DSDP) and $(\mathrm{SDP_1})$ are modeled by `CVX 1.2` and solved by `SeDuMi` 1.2;
  - The SOCP problem $(\mathrm{SOCP_d})$ is solved by MOSEK 6.0;
  - The convex QP problem (QP) is solved by the QP solver `quadprog` in Matlab Optimization Toolbox.

Test Problems

- The parameters in the test problems are randomly generated (this could lead to over-optimistic computation time).
- Test problem 1 (portfolio selection):

$$
\begin{aligned}
(\mathrm{MV}) \qquad \min \quad & -r^{\mathrm{T}}x + \frac{1}{2}(x - x^B)^{\mathrm{T}}\Sigma(x - x^B) + \sum_{i=1}^{n} c_i(x_i - x_i^0)^2, \\
\text{s.t.} \ & \Big|\sum_{i\in S_k}(x_i - x_i^B)\Big| \le \epsilon_k, \ k = 1, \ldots, m, \\
& e^{\mathrm{T}}x = 1, \ x \ge 0, \\
& |\mathrm{supp}(x)| \le K, \\
& x_i \ge \alpha_i, \ i \in \mathrm{supp}(x),
\end{aligned}
$$

where the parameters are simulated weekly returns (Gaussian distribution).

- Test problem 2:

$$
(\mathrm{P}_1) \qquad \begin{aligned} \min\ & \frac{1}{2}x^{\mathrm{T}}(H^{\mathrm{T}}H + \mathrm{diag}(\varrho))x - c^{\mathrm{T}}x \\ \mathrm{s.t.}\ & Ax \geq b,\ 0 \leq x \leq u, \\ & |\mathrm{supp}(x)| \leq K, \\ & x_i \geq \alpha_i,\ i \in \mathrm{supp}(x), \end{aligned}
$$

where the parameters are uniformly distributed in some intervals.

Table: Comparison results for test problem 1 with $K = \frac{n}{2}$ and $m = 5$

n	(MIQP_0)			(DSDP)	$(\mathrm{MIQP}_\mathrm{d})$		
	time	nodes	rel. error (%)	time	time	nodes	rel. error (%)
20	27.46	3111	0.00	0.72	4.01	41	0.00
30	1415.77	108847	0.85	1.09	14.01	234	0.00
40	1800.00	82385	2.37	1.69	27.79	591	0.01
60	1800.00	35555	0.87	3.19	172.18	1549	0.01
80	1800.00	21850	0.73	5.29	1519.76	8759	0.07
100	1800.00	11023	0.84	8.27	1800.00	6474	0.23

Table: Average improv ratio of lower bounds for test problem 2 with $m = 10,\ K = \lfloor \frac{n}{4} \rfloor$

n	improv. ratio (%)		CPU time (seconds)		
	$v(\mathrm{SOCP_d})$	$v(\mathrm{SDP_1})$	$v(\mathrm{QP_0})$	$v(\mathrm{SCOP_d})$	$v(\mathrm{SDP_1})$
100	33.19	61.48	0.02	0.32	6.34
200	18.01	38.15	0.05	1.60	36.12
300	10.19	24.21	0.11	4.33	117.65
400	7.69	20.59	0.22	9.77	258.93
500	6.03	17.83	0.39	19.75	528.83

Table: Comparison results for problem 2 with $m = \lfloor \frac{n}{4} \rfloor$ $K = \lfloor \frac{n}{4} \rfloor$

n	(MIQP_0)			SDP	$(\mathrm{MIQP}_\mathrm{d})$		
	time	nodes	rel. gap(%)	time	time	nodes	rel. gap(%)
20	10.14	1124	0.00	0.70	8.13	459	0.00
30	186.02	15526	0.00	1.04	92.04	4431	0.00
40	1588.93	78007	16.24	1.53	752.69	21876	1.66
60	1800.00	37174	55.65	2.72	1800.00	24294	36.24
80	1800.00	18097	60.34	4.44	1800.00	11829	41.98

Discussions

- Lagrangian relaxation technique has been successfully applied to many NP-hard integer and combinatorial optimization problems (Fisher 1981) to generate tight dual bounds in a branch-and-bound framework or construct approximate feasible solution.
- Subgradient methods are commonly used to search the optimal multipliers and dual value. In some cases (when the relaxation is "easy" to solve), it is possible to reduce the dual problem to a polynomially solvable convex formulation such as LP, SDP, SOCP ...
- We have used the matrix decomposition $Q = (Q - \mathrm{diag}(d)) + \mathrm{diag}(d)$ and Lagrangian decomposition technique to generate SDP relaxation and new MIQP reformulation for cardinality QP.

Further Study

- Extend the idea of SDP reduction to Lagrangian dual formulations for other integer and combinatorial optimization problems.
- Develop Lagrangian heuristics for finding suboptimal solutions of large-scale cardinality QP.
- Lagrangian decomposition and SDP relaxations for probabilistic constrained QP:

$$\text{(CCQP)} \qquad \begin{aligned} \min\ & \frac{1}{2}x^T Qx + c^T x \\ \text{s.t. } & \text{Prob}(\xi^T x \geq b) \geq 1-\epsilon, \\ & x \in X. \end{aligned}$$

- Question: How to construct tight convex relaxations and efficient MQIP reformulation to(CCQP)?