On the Uniqueness and Structure of Solutions to the Equation and System Arising from Chern-Simons Models

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• The talk is based on a joint work with Jann-Long Chern, Chang-Shou Lin and Yong-Li Tang.

In this talk, we consider the following equation and system

$$\Delta u + \frac{4}{\kappa^2} e^u (1 - e^u) = 4\pi \sum_{j=1}^{N} \delta_{p_j} \text{ in } \mathbf{R}^2, \tag{1}$$

and

$$\begin{cases} \Delta u + \lambda e^v (1 - e^u) = 4\pi \sum_{s=1}^{N'} \alpha'_s \delta_{p'_s} \text{ in } \mathbf{R}^2, \\ \Delta v + \lambda e^u (1 - e^v) = 4\pi \sum_{s=1}^{N''} \alpha''_s \delta_{p''_s} \text{ in } \mathbf{R}^2, \end{cases} \tag{2}$$

where $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$, κ and λ are positive constants, N, N' and N'' are positive constants which are called the vortex numbers, $\alpha'_s > 0$ and $\alpha''_s > 0$ are constants, and δ_p is the Dirac measure at p.

Remark. Equation(1) and System(2) arise respectively from a relativistic Abelian Chern-Simons model with one and two Higgs particles on plane.

Reference.

- H. Chan, C.-C. Fu, and C.-S. Lin, Comm. Math. Phys. **231**(2002).
- C.-S. Lin, A. C. Ponce, and Y. Yang, J. Funct. Anal. **247**(2007), 289–350.

(A) How about the existence and uniqueness of solutions for Eq.(1) and System(2) ?

(B) How to classify the solutions for Eq.(1) and System(2) respectively ?

Solution Types of Chern-Simons Equation

$$\Delta u + \frac{4}{\kappa^2} e^u(1 - e^u) = 4\pi \sum_{j=1}^{N} \delta_{p_j} \text{ in } \mathbf{R}^2$$

Now, we consider the following two types of solutions, .

$$u(x) \to 0 \quad \text{as} \quad |x| \to \infty : \textbf{Topological Solution}$$

and

$$u(x) \to -\infty \quad \text{as} \quad |x| \to \infty : \textbf{Non-Topological Solution}$$

.

- By Maximum Principle, topological solution and non-topological solution are negative in $\mathbf{R}^2 \backslash \{p_1 \ldots p_N\}$.

Flux of Chern-Simons Equation

For any entire solution u of (1), define the flux

$$\Phi = \frac{2}{\kappa^2} \int_{\mathbf{R}^2} e^u(1 - e^u)dx.$$

- If u is a topological solution of (1), then $\Phi = 2\pi N$. [Joel Spruck and Yang 1995]
- If u is a non-topological solution of (1), then $\Phi > 4\pi(N+1)$. [Joel Spruck and Y.-S. Yang 1995]
- From [C.-S. Lin and K.-S. Cheng 1997], if u is a negative solution of (1), then

$$\frac{2}{\kappa^2} \int_{\mathbf{R}^2} e^u(1 - e^u)dx < \infty.$$

Furthermore, $u(x)$ is a topological solution or non-topological Solution.[C.-S. Lin, A.-C. Ponce and Y.-S. Yang 2008]

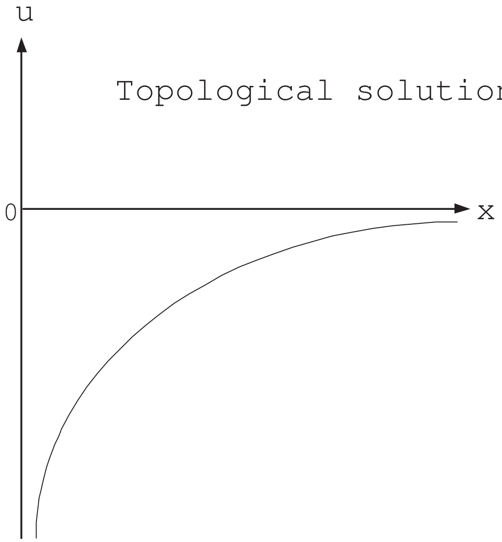


Figure 1.

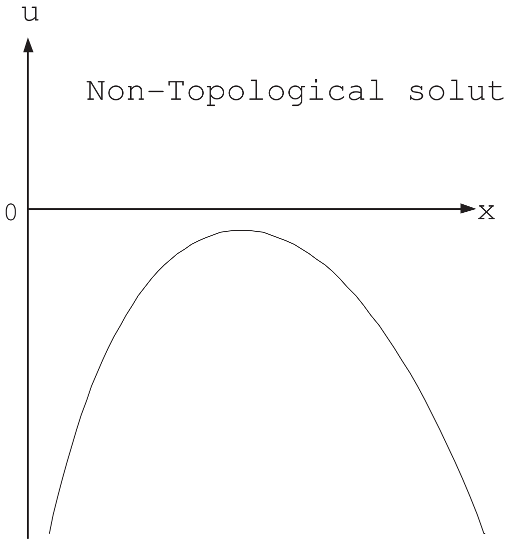


Figure 2.

Question 1

(i) Does the topological solution exist ?

(ii) For any given number $\Phi > 4\pi(N+1)$, does (1) possess a non-topological solution u with the given Φ as its flux ?

Existences of Topological and Non-topological Solutions for (1)

- (1) possesses a topological solution u. [Spruck and Yang 1995]
- There is a sequence $\{\Phi_i\}_{i=1}^{\infty}$ which is close to $4\pi(N+1)$ such that the non-topological solution of (1) with the given Φ_i as its flux can be obtained. [Chae-Imanuvilov, 2000]
- Suppose $\Phi > 8\pi N, p_i \neq p_j\ \forall i \neq j$ and for each $j = 1, 2, \cdots, N$,

$$\sum_{k \neq j}^{N} \log(|p_j - p_k|) \ \text{ is independent of } j.$$

  Then $\exists \kappa_* > 0$ such that $\forall 0 < \kappa \leq \kappa_*$, (1) possesses a non-topological solution u with the given Φ as its flux. [Chan-Fu-Lin, 2002]
- For any $\Phi > 4\pi(N+1)$,(1) possesses a non-topological solution u with the given Φ as its flux if $p_1 = p_2 = \cdots = p_N$. [Chan-Fu-Lin, 2002]

Question 2

(i) Is the topological solution unique ?

(ii) For any given number $\Phi > 4\pi(N+1)$, does (1) possess a unique non-topological solution u with the given Φ as its flux ?

Uniqueness of Topological and Non-topological Solutions for (1)

If $p_1 = p_2 = \cdots = p_N$, then

(1) The topological solution of (1) is unique and radial. [Chen-Hasting-McLeod-Yang, 1994]

(2) For any given number $\Phi > 4\pi(N+1)$, (1) possesses a unique radial solution u with the given Φ.[Chan-Fu-Lin, 2002]

Radial Solutions of (1) with the Same Vorticities

In order to investigate the structure of all radial solutions of (1) with the same vorticities, we consider the following ODE

$$u''(r) + \frac{1}{r}u'(r) + e^{u(r)}(1 - e^{u(r)}) = 0, \ \ r > 0 \qquad (D)$$

with the initial value

$$u(r) = 2N \log r + \alpha + o(1) \text{ as } r \to 0. \qquad (I)$$

Uniqueness of Solutions of Dirichlet Problem to (1) with the Same Vorticities

- All solutions of Dirichlet problem to (1) with the same vortex are radially symmetry.

Theorem (Chern-Chen-Tang, 2008)

For any given $R > 0$. Then the solution u_R of (D) and (I) with $u_R = 0$ on ∂B_R is unique and the following properties are valid.

(i) *u_R is a blowup solution.*

(ii) *There exists a strictly monotone C^1 functions $\alpha_\cdot(0,\infty) \to \mathbf{R}$ such that*

$$\alpha'(R) < 0 \ \ \forall R \in (0,\infty) \quad \textit{and} \quad \lim_{R\to\infty} \alpha(R) = \alpha_T$$

where $u(R,\alpha(R)) = 0$ and $u(r,\alpha_T)$ is a topological solution of (D).

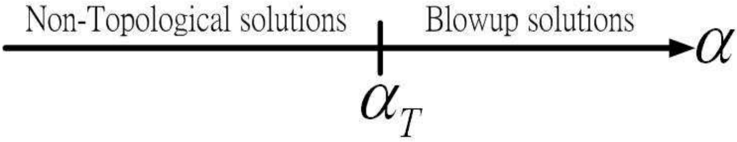


Figure 4.

Flux

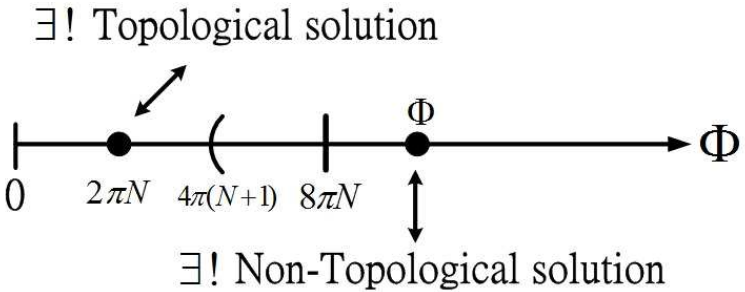


Figure 5.

We consider the Chern-Simmons system as follows:

$$\begin{cases} \Delta u + \lambda e^v(1 - e^u) = 4\pi \sum_{s=1}^{N'} \alpha'_s \delta_{p'_s} \text{ in } \mathbf{R}^2, \\ \Delta v + \lambda e^u(1 - e^v) = 4\pi \sum_{s=1}^{N''} \alpha''_s \delta_{p''_s} \text{ in } \mathbf{R}^2. \end{cases} \tag{3}$$

Interesting Problems:

(i) Does System(3) possess a topological or non-topological solution ? If yes, is the topological unique ?

(ii) How about the respective **Flux** of each solution ?

(iii) For any given Φ, does (3) possess a non-topological solution with Φ as its flux ?

(iv) How about other solutions ? Can we classify all solutions ?

Topological and Negative Solutions for (3)

Now we consider the solution pair (u, v) of (3) satisfying the boundary condition:

(i) $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0$ (**Topological Solution**)

(ii) $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = -\infty$

$\Rightarrow$ By Maximum Principle, we have

$$u(x) < 0 \text{ and } v(x) < 0$$

$\forall x \in \mathbf{R}^2 \setminus \{p_1', p_1'' \ldots p_{N'}', p_{N''}''\}$.

The general existence about topological solution has been proved by [Lin-Ponce-Yang] recently:

Theorem (Lin-Ponce-Yang, 2008 JFA)

For any given sets $\{p'_1, \cdots, p'_{N'}\}$ *and* $\{p''_1, \cdots, p''_{N''}\}$ *and* $\alpha'_s,\ \alpha''_s > 0$*, the Eq.(3) possesses a topological solution* (u, v)*.*

- According to Theorem above, it is natural to ask the question about the uniqueness of topological solutions for Eq.(3).

Uniqueness of Topological Solution of (3)

We consider a topological solution (u, v) for the case $N' = N'' = 1$ and p_1' and p_1'' to be the origin O. Then (u, v) satisfies

$$\begin{cases} \Delta u + e^v(1 - e^u) = 4\pi N_1 \delta(0) \\ \Delta v + e^u(1 - e^v) = 4\pi N_2 \delta(0) \end{cases}, \tag{4}$$

in $\mathbf{R}^2$, with the boundary condition

$$u(x) \to 0, \ \ v(x) \to 0 \text{ as } |x| \to \infty. \tag{5}$$

Theorem (Theorem 1. Chern-Chen-Lin, 2010 CMP)

Equation (4) possesses one and only one topological solution pair (u, v). Furthermore, u and v are radially symmetric.

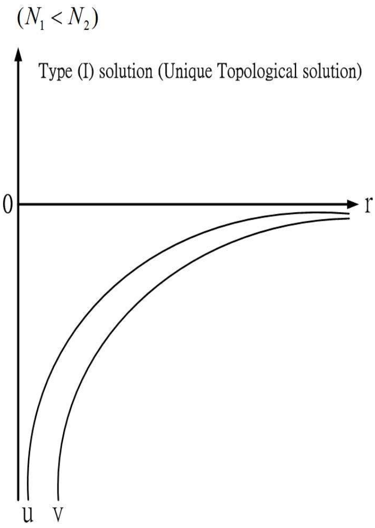


Figure 6.

Radial Solutions of Two Higgs Particles with the Same Vorticities

In order to investigate the structure of all radial solutions of (4), we consider the following ODE system

$$\begin{cases} u''(r) + \frac{1}{r}u'(r) + e^{v(r)}(1 - e^{u(r)}) = 0, \\ v''(r) + \frac{1}{r}v'(r) + e^{u(r)}(1 - e^{v(r)}) = 0, \end{cases} \quad r > 0 \tag{6}$$

with the initial value

$$\begin{cases} u(r) = 2N_1 \log r + \alpha_1 + o(1), \\ v(r) = 2N_2 \log r + \alpha_2 + o(1) \end{cases} \quad \text{as } r \to 0^+. \tag{7}$$

Existence and Uniqueness of the Solution of (4) with $u(R) = v(R) = 0$

Theorem (Theorem 2. Chern-Chen-Lin-Tang, 2009 JDE)

For any positive numbers N_1, N_2 and R, (4) possesses one and only one solution (u_R, v_R) with $u(R) = v(R) = 0$ and the following properties are valid.

(i) *u_R and v_R are radially symmetric satisfying*

$$u_R(r) < 0,\ \ v_R(r) < 0,\ \ u'_R(r) > 0 \text{ and } v'_R(r) > 0\ \forall r \in (0, R].$$

(ii) *For each $r \in (0, R]$ we have*

$$\begin{cases} u_R(r) > v_R(r) \text{ if } N_1 < N_2, \\ u_R(r) = v_R(r) \text{ if } N_1 = N_2, \\ u_R(r) < v_R(r) \text{ if } N_1 > N_2. \end{cases}$$

Theorem

(iii) *There exist two strictly monotone C^1 functions* $\gamma_1, \gamma_2 : (0,\infty) \to \mathbf{R}$ *such that*

$$\begin{cases} D = \{\alpha(R) | \alpha(R) = (\gamma_1(R), \gamma_2(R)) \ \forall R \in (0,\infty)\}, \\ \gamma_1'(R) < 0 \text{ and } \gamma_2'(R) < 0 \ \forall R \in (0,\infty) \ \textit{and} \\ \lim_{R\to\infty} \alpha(R) = (\alpha_{10}, \alpha_{20}) = \alpha_0, \end{cases}$$

where

$$D = \{\alpha | (u(r,\alpha), v(r,\alpha)) \ \textit{is a solution of}(4) \ \textit{with } u(R){=}v(R){=}0 \textit{ for some } R > 0\}.$$

and $(u(r,\alpha_0), v(r,\alpha_0))$ *is a topological solution of (4).*

The asymptotic behaviors of all entire solutions for (6) are as follows. Here solutions are not necessarily negative in $\mathbf{R}^2$.

Theorem (Theorem 3. Chern-Chen-Lin, 2010 CMP)

Let (u, v) be an entire solution of $(6) - (7)$. Then (u, v) satisfies one of the following behaviors.

(A) $\lim_{r\to\infty} u(r) = 0$ *and* $\lim_{r\to\infty} v(r) = 0$;

(B) $\lim_{r\to\infty} u(r) = -\infty$ *and* $\lim_{r\to\infty} v(r) = -\infty$;

(C) $\lim_{r\to\infty} (u(r), \frac{v(r)}{r^2}) = (-c_u, -\frac{e^{-c_u}}{4})$ *for some* $c_u > 0$;

(D) $\lim_{r\to\infty} (\frac{u(r)}{r^2}, v(r))) = (-\frac{e^{-c_v}}{4}, -c_v)$ *for some* $c_v > 0$;

(E) $\lim_{r\to\infty} u(r) = \infty, \quad \lim_{r\to\infty} v(r) = -\infty$;

(F) $\lim_{r\to\infty} u(r) = -\infty, \quad \lim_{r\to\infty} v(r) = \infty$.

Flux

- For an entire solution (u, v) of (6)-(7), we define the **flux pair**

$$\beta_1 = \int_0^\infty re^v(1 - e^u)\ dr,\ \ \beta_2 = \int_0^\infty re^u(1 - e^v)\ dr. \qquad (8)$$

- If (u, v) is a topological of (6), then its flux pair satisfies $\beta_1 = 2N_1,\ \ \beta_2 = 2N_2$.
- A solution (u, v) is called a non-topological solution of (6) if $(u(r), v(r)) \to -\infty$ as $r \to \infty$, and its corresponding flux $\beta_1 < \infty, \beta_2 < \infty$.

All Types of Solutions for (6)

According to the behaviors at ∞, all entire solutions of (6) can be classified into the following five types.

Type (I): $\lim_{r\to\infty}(u(r),v(r)) = (0,0)$, i.e., (u,v) is a topological solution.

Type (II): $\lim_{r\to\infty}(u(r),v(r)) = (-\infty,-\infty)$ with $\beta_1 < \infty$ and $\beta_2 < \infty$, i.e.,
(u,v) is a non-topological solution.

Type (III): $\lim_{r\to\infty} u(r) = -\infty$, $\lim_{r\to\infty} v(r) = -\infty$, and either
$2N_1 < \beta_1 \le 2N_1 + 2$, $\beta_2 = \infty$ or
$\beta_1 = \infty$, $2N_2 < \beta_2 \le 2N_2 + 2$.

Type (IV): $\lim_{r\to\infty}(u(r),v(r)) = (-c_u,-\infty)$ or $(-\infty,-c_v)$ for some constants $c_u > 0$ and $c_v > 0$.

Type (V): $\lim_{r\to\infty}(u(r),v(r)) = (+\infty,-\infty)$ or $(-\infty,+\infty)$.

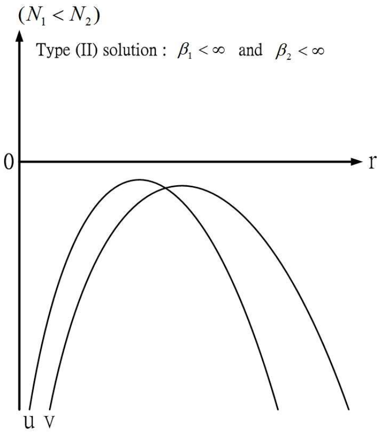


Figure 7.

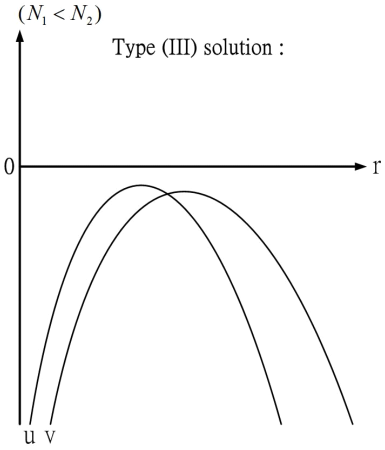


Figure 8.

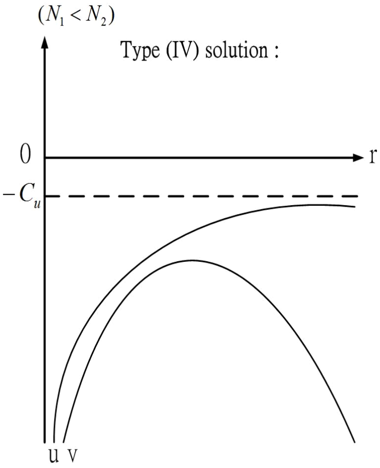


Figure 9.

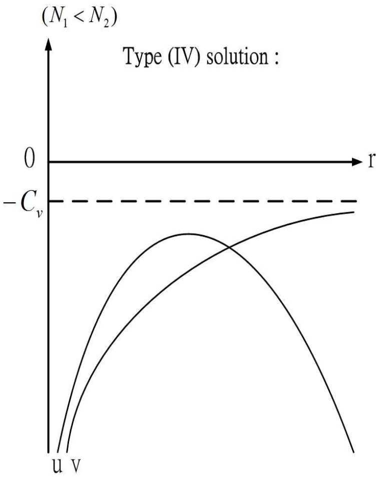


Figure 10.

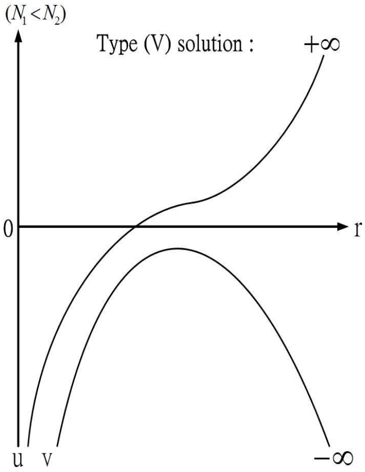


Figure 11.

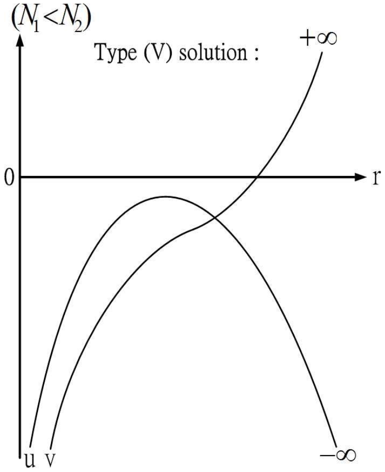


Figure 12.

Let $\alpha = (\alpha_1, \alpha_2)$, and $(u(r,\alpha), v(r,\alpha))$ denote the solution of (6)-(7). According to the behavior of (u, v), the set of initial data could be classified into the following regions.

- $\Omega = \{\alpha | (u(r,\alpha), v(r,\alpha))$ is a solution with $\lim_{r\to\infty} (u(r,\alpha), v(r,\alpha)) = (-\infty, -\infty)\}$
- $T = \{\alpha | (u(r,\alpha), v(r,\alpha))$ is the unique topological solution$\}$
- $\Omega_{NT} = \{\alpha | (u(r,\alpha), v(r,\alpha))$ is a non-topological solution$\}$

- $S_u = \{\alpha | (u(r,\alpha), v(r,\alpha))$ is a Type (IV) solution with $\lim_{r\to\infty} u(r) = -c_u\}$
- $S_v = \{\alpha | (u(r,\alpha), v(r,\alpha))$ is a Type (IV) solution with $\lim_{r\to\infty} v(r) = -c_v\}$
- $W_u = \{\alpha | (u(r,\alpha), v(r,\alpha))$ is a Type (V) solution with $\lim_{r\to\infty} u(r) = \infty\}$
- $W_v = \{\alpha | (u(r,\alpha), v(r,\alpha))$ is a Type (V) solution with $\lim_{r\to\infty} v(r) = \infty\}$.

Structure of Radial Solutions for (6)

Theorem (Theorem 4. Chern-Chen-Lin, 2010 CMP)

Both Ω and Ω_{NT} are non-empty and open simply connected. Furthermore, all sets $\Omega \backslash \Omega_{NT}, \Omega_u, \Omega_v, S_u, S_v, W_u$ and W_v are non-empty, and the following statements are valid.

(i) *$\Omega = \Omega_v \bigcup \Omega_{NT} \bigcup \Omega_u$ is a non-empty and simply connected set, where*

$$\Omega_u = \{\alpha \in \Omega | (u(r,\alpha), v(r,\alpha)) \text{ is a Type (III) solution with } \beta_1 < \infty\},$$

$$\Omega_v = \{\alpha \in \Omega | (u(r,\alpha), v(r,\alpha)) \text{ is a Type (III) solution with } \beta_2 < \infty\}.$$

Theorem (4.Cont.)

(ii) $\partial\Omega = S_u \bigcup T \bigcup S_v$ *and* $\overline{S}_u \bigcap \overline{S}_v = \partial\Omega \bigcap \partial\Omega_{NT} = T$.

(iii) *For any* $\alpha \in \Omega_{NT}$ *the corresponding* (β_1, β_2) *satisfies*

$$(\beta_1 - 2(N_1 + 1))(\beta_2 - 2(N_2 + 1)) > 4(N_1 + 1)(N_2 + 1). \qquad (9)$$

(iv) W_u *is open. Furthermore, for each* $(\theta, \eta) \in S_u$ *there exists* $\epsilon > 0$ *such that* $(\alpha_1, \eta) \in W_u$ $\forall \theta < \alpha_1 < \theta + \epsilon$.

(v) W_v *is open. Furthermore, for each* $(\mu, \nu) \in S_v$ *there exists* $\delta > 0$ *such that* $(\mu, \alpha_2) \in W_v$ $\forall \nu < \alpha_2 < \nu + \delta$.

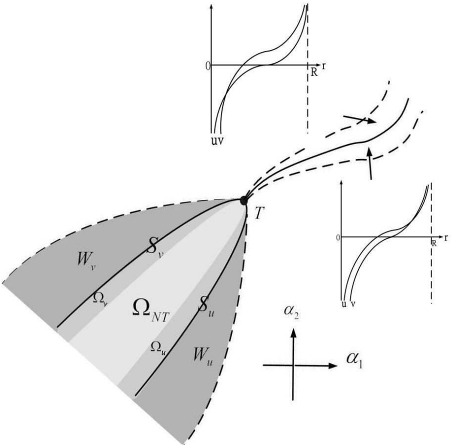


Figure 13. Structure of Radial Solutions for Two Higgs Particles Model

The set of initial data for Non-Topological solutions

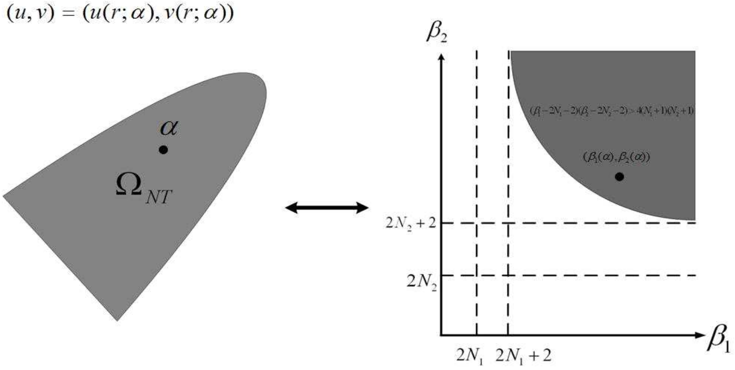


Figure 14.

We prove the uniqueness theorem by contradiction.
Suppose that for some pair (N_1^0, N_2^0), Eq.(6) possesses at least two topological solutions. Without loss of generality, we may assume $0 \leq N_1^0 < N_2^0$. Let

$$N_1^* = \inf\{0 \leq N_1 | (4) \text{ possesses a unique topological solution for all } (\hat{N}_1, N_2^0) \text{ where } N_1 \leq \hat{N}_1 \leq N_2^0\}.$$

Clearly, $N_1^* \geq N_1^0$. By the definition of N_1^*, there are two sequences of solutions $(u_k, v_k), (u_k^*, v_k^*)$ of (4) with (N_1^k, N_2^0) such that $N_1^k \downarrow N_1^*$. We divide the proof into the following steps.

Step 1. We claim that there exists a subsequence of (U_k, V_k) such that it converges to (U, V) in $C^2(\overline{B_R(0)}) \times C^2(\overline{B_R(0)})$ for any $R > 0$, where
$(u(r), v(r)) = (U(r) + 2N_1^* \log r, V(r) + 2N_2^0 \log r)$ and
$(u_k(r), v_k(r)) = (U_k(r) + 2N_1^k \log r, V_k(r) + 2N_2^0 \log r)$ are topological solutions of (4).

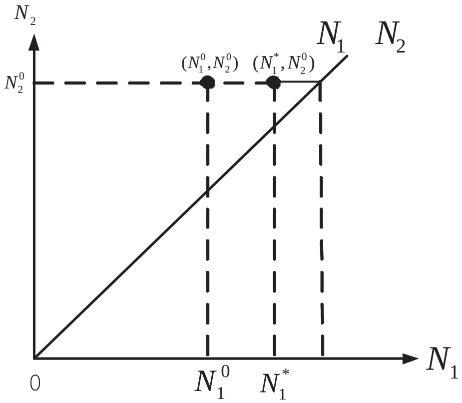
Step 2. (Non-Degenerate Property) Let (u, v) be a topological solution of (4). Then the corresponding linearized eq. is non-degenerate, i.e.,

$$\begin{cases} \Delta A + e^v(1 - e^u)B - e^{u+v}A = 0 \\ \Delta B + e^u(1 - e^v)A - e^{u+v}B = 0 \end{cases} \tag{10}$$

does not have any bounded non-zero solution (A, B) in $[0, \infty)$.

Step 3. We claim that

(∗) suppose (u_0, v_0) is a topological solution of (1.15) with respect to (N_1', N_2'). Let $U_0(r) = u_0(r) - 2N_1' \log r$ and $V_0(r) = v_0(r) - 2N_2' \log r$. Then there is a neighborhood B of (N_1', N_2') such that for any pair of (N_1, N_2) in B, there exists corresponding (U, V) with respect to (N_1, N_2), which is close to (U_0, V_0) in $C^2(\overline{B_R(0)}) \times C^2(\overline{B_R(0)})$ for any $R > 0$, where $(u(r), v(r)) = (U(r) + 2N_1 \log r, V(r) + 2N_2 \log r)$ is a topological solution of (4) with respect to (N_1, N_2).



N_1^* N_1^0

Figure 15.

Step 4. By Lemma above, (u_k, v_k) and (u_k', v_k') converge to (u, v) due to the fact that (4) has only one topological solution at (N_1^*, N_2^0). W.l.o.g., we may assume $|(u_k - u_k')(x_k)| = ||u_k - u_k'||_{L^\infty} \geq ||v_k - v_k'||_{L^\infty}$ $\forall k$. Set $A_k = \frac{(u_k - u_k')}{||u_k - u_k'||_{L^\infty}}$ and $B_k = \frac{(v_k - v_k')}{||u_k - u_k'||_{L^\infty}}$. Then A_k, B_k satisfy

$$\begin{cases} \Delta A_k + e^{\eta_k(x)}(1 - e^{u_k})B_k - e^{\xi_k(x)+v_k}A_k = 0 \\ \Delta B_k + e^{\xi_k(x)}(1 - e^{v_k})A_k - e^{\eta_k(x)+u_k}B_k = 0, \end{cases}$$

where $\xi_k(x) \in (u_k(x), u_k'(x))$ and $\eta_k(x) \in (v_k(x), v_k'(x))$. Since for any fixed k, $u_k(x) \to 0, v_k(x) \to 0$ as $x \to \infty$, we can obtain that the maximum points x_k are bounded. Thus there exists subsequence of (A_k, B_k) which converges to A and B in $C^2(\mathbf{R}^2)$ respectively, where (A, B) satisfies the linearized system of (12). Since A and B are bounded and not all zero in $\mathbf{R}^2$, by **Step 2**, we have $A \equiv 0$ and $B \equiv 0$, a contradiction. This completes the proof of uniqueness.

By the method of moving planes, we see that $u(r)$ and $v(r)$ are radially symmetric where $(u(r), v(r))$ is a solution of (5) on $B_R(O)$ and satisfies $u(R) = v(R) = 0$. From the maximum principle, it is easy to obtain the results (i) and (ii).

(iii)Let $U(r) = u(r) - 2N_1 \log r$, $V(r) = v(r) - 2N_2 \log r$, and

$$\begin{cases} \phi_i(r) = \dfrac{\partial U(r,\alpha)}{\partial \alpha_i}, \\ \psi_i(r) = \dfrac{\partial V(r,\alpha)}{\partial \alpha_i}. \end{cases} \tag{11}$$

Then $(\phi_i, \psi_i), i = 1, 2$, satisfy the corresponding linearized equations

$$\begin{cases} \Delta\phi_i - e^{u+v}\phi_i + e^v(1 - e^u)\psi_i = 0, \ \ r \in [0, R], \\ \Delta\psi_i - e^{u+v}\psi_i + e^u(1 - e^v)\phi_i = 0, r \in [0, R], \\ \phi_1(0) = 1 = \psi_2(0), \phi_2(0) = 0 = \psi_1(0), \\ \phi_i'(0) = 0 = \psi_i'(0). \end{cases} \tag{12}$$

Step 1.

$$\begin{cases} \phi_1(r) > 0, \phi_1'(r) > 0, \ \phi_2(r) < 0, \phi_2'(r) < 0, \\ \psi_1(r) < 0, \psi_1'(r) < 0, \ \psi_2(r) > 0, \psi_2'(r) > 0 \end{cases} \forall r \in (0, R].$$

Step 2.

$$\det \begin{pmatrix} \phi_1(r) & \phi_2(r) \\ \psi_1(r) & \psi_2(r) \end{pmatrix} \neq 0$$

for $r \in [0, R_0]$. Furthermore, one of $\phi_1(R) + C\phi_2(R)$ and $\psi_1(R) + C\psi_2(R)$ is positive for any $C > 0$.

Define the function F by

$$\begin{aligned} F(r,\alpha_1,\alpha_2) &= (u(r,\alpha_1,\alpha_2), v(r,\alpha_1,\alpha_2))^T \\ &\equiv (F_1(r,\alpha_1,\alpha_2), F_2(r,\alpha_1,\alpha_2))^T \end{aligned} \tag{13}$$

for $r > 0$ and $(\alpha_1, \alpha_2) \in \mathbf{R}^2$. Denote the zero set of F by

$$\Theta = \{(r,\alpha_1,\alpha_2) | F(r,\alpha_1,\alpha_2) = (0\ 0)\}\,. \tag{14}$$

If $(R_0, \alpha_1^0, \alpha_2^0) \in \Theta$, then we obtain that there exist $\varepsilon = \varepsilon(R) > 0$ and a unique C^1 function curve $(\gamma_1, \gamma_2) : (R_0 - \varepsilon, R_0 + \varepsilon) \to \mathbf{R}^2$ such that $(R, \gamma_1(R), \gamma_2(R)) \in \Theta \ \forall R \in (R_0 - \varepsilon, R_0 + \varepsilon)$ by Implicit Function Theorem.
Because of

$$\begin{cases} u'(R, \gamma_1(R), \gamma_2(R)) + \phi_1(R)\gamma_1'(R) + \phi_2(R)\gamma_2'(R) = 0 \\ v'(R, \gamma_1(R), \gamma_2(R)) + \psi_1(R)\gamma_1'(R) + \psi_2(R)\gamma_2'(R) = 0 \end{cases} \tag{15}$$

for $R > 0$, we complete this proof.

Thanks for Your Attention