Contact geometry and the 3-body problem

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Consider two primaries, let us say the Earth with mass $1-\mu$ and the Moon with mass $\mu \in [0,1]$, moving around each other in a circular orbit.

$$E(t) = (-\mu \cos t, \mu \sin t) \qquad M(t) = ((1-\mu)\cos t, -(1-\mu)\sin t)$$

Then the movement of third massless particle, say a satellite, is determined by the following Hamiltonian

$$H(q,p) = \frac{1}{2}|p|^2 - \frac{1-\mu}{|q-E(t)|} - \frac{\mu}{|q-M(t)|}.$$

This gives a (time-dependent) dynamical system via the Hamilton equations:

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \qquad \dot{q}_j = \frac{\partial H}{\partial p_j}$$

Surprisingly, Jacobi found an integral by using a rotating coordinate system. This is the following Hamiltonian,

$$H_J(q,p) = \frac{1}{2}|p|^2 + q_1 p_2 - q_2 p_1 - \frac{1-\mu}{|q+\mu|} - \frac{\mu}{|q-(1-\mu)|},$$

which is no longer time-dependent.

For $\mu \neq 0, 1$, this system is not integrable. Note that the H_J is preserved by the Hamilton equations. Also note that the equations are singular: satellite-Earth or satellite-Moon collisions are possible: the hypersurfaces $H_J = E$ are not compact.

With respect to the symplectic form

$$\omega = dp \wedge dq$$

the Hamilton equations can be seen as the flow of the Hamilton vector field X_{H_J} for H_J,

$$i_{X_{H_J}} \omega = -dH_J.$$

The dynamics change dramatically by changing the energy level. This is particularly clear if the energy level passes a critical point of the Hamiltonian H_J. This Hamiltonian has five critical points, called Lagrange points.

- $H = E < H(L_1)$: three connected component: close to Earth, close to the Moon, and the region near infinity. There is not enough energy to go from the Earth to the Moon.

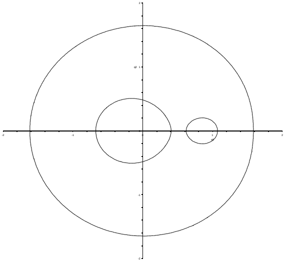


Figure: Hill's region for $H = E < H(L_1)$: the q-coordinates of the satellite will always stay in this region

- $H = E \in]H(L_1), H(L_2)[$: two connected components. An Earth-Moon component, and the region near infinity. There is sufficient energy to go from the Earth to the Moon, but not enough to escape to infinity.

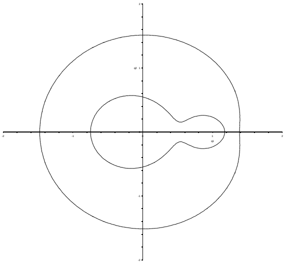


Figure: Hill's region for $H = E \in]H(L_1), H(L_2)[$

- $H = E \in]H(L_2), H(L_3)[$: only one connected component; one can escape to infinity via the Moon.

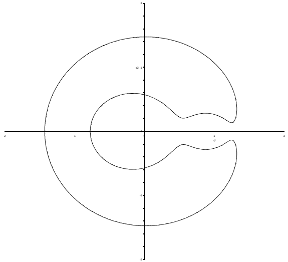


Figure: Hill's region for $H = E \in]H(L_2), H(L_3)[$

- $H = E \in]H(L_3), H(L_4) = H(L_5)[$: one connected component

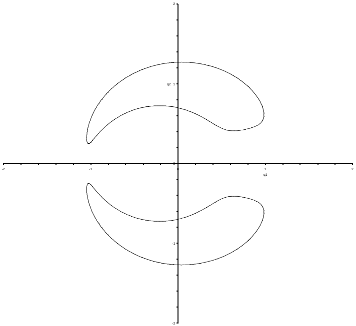


Figure: Hill's region for $H = E \in]H(L_3), H(L_4) = H(L_5)[$

- $H = E > H(L_4) = H(L_5)$: one connected component

Regularizing collisions: the Kepler problem

$$H_K = \frac{1}{2}|p|^2 - \frac{1}{|q|}.$$

Consider the level set $H_K = -c$. The Hamiltonian

$$K = |q|(H_K + c) = \frac{1}{2}(|p|^2 + 2c)|q| - 1$$

on the level set $K = 0$ has the same dynamics (but different parametrization) as $H_K = -c$. Finally, the dynamics do not change if we use

$$Q = \frac{1}{4}(|p|^2 + 2c)^2|q|^2$$

on the level set $Q = 1$.

The Hamiltonian

$$Q = \frac{1}{4}(|p|^2 + 2c)^2 |q|^2$$

can also be seen as the Hamiltonian for the geodesic flow on $S_g T^* S^2$ after stereographic projection. This allows us to compactify level sets of H_K to $\mathbb{R}P^3$. Also, we can see the dynamics of the Kepler problem as geodesics on the round S^2.

A similar regularization works for the 3-body problem. Note we need to regularize both near the Earth and the Moon. The dynamics are much more complicated than those of the Kepler problem.

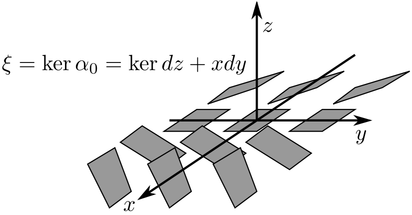
Contact manifolds

Definition

A **contact manifold** $(M, \xi = \ker \alpha)$ is a $2n+1$-dimensional manifold M with a maximally non-integrable field of hyperplanes ξ, i.e.

$$\alpha \wedge d\alpha^n \neq 0.$$

ξ is called the **contact structure** and α a **contact form**.



- Dynamical systems. Given a contact form α on M, we can define the Reeb field by

$$i_R d\alpha = 0, \;\; i_R \alpha = 1$$

- The Reeb flow on $S_g T^* M$ is the geodesic flow.
- Dynamics depend on the choice of form α.

Remark

The later role of the contact condition is taming the dynamics.

Definition

Two contact manifolds $(M, \xi = \ker \alpha)$ and $(N, \eta = \ker \beta)$ are said to be **contactomorphic** if there is a diffeomorphism $\psi : M \to N$ such that

$$\psi^* \beta = f \alpha$$

for a non-zero function f.

This means that ψ sends contact planes to contact planes

Theorem (Darboux)

Let $(M, \xi = \ker \alpha)$ be a contact manifold. Every $p \in M$ has a neighborhood U such that $(U, \alpha) \cong (\mathbb{R}^{2n+1}, \alpha_0)$.

Contact manifolds have no local invariants.

Definition

A **symplectic manifold** (W^{2n}, ω) is a smooth manifold with a closed, non-degenerate 2-form ω, i.e. $\omega^n \neq 0$.

Examples: $(\mathbb{R}^{2n}, \omega = d\vec{x} \wedge d\vec{y})$ and $(T^*M, d\lambda_{can} = dp \wedge dq)$ for any smooth manifold M.

Definition

Let (W, ω) be a symplectic manifold and $M \subset W$ a hypersurface. A vector field X defined on a neighborhood of M is called **Liouville field** for M if X is transverse to TM and $\mathcal{L}_X \omega = \omega$

Proposition

If X is Liouville for $M \subset (W, \omega)$, then $(M, i_X \omega)$ is contact.

We see that $(S^{2n-1}, \vec{x} d\vec{y} - \vec{y} d\vec{x})$ and $(ST^*M, \lambda_{can} = pdq)$ are contact.

Theorem (Albers, Frauenfelder, Paternain, vK)

The (regular) compact level sets of the regularized, planar restricted 3-body problem are contact up to energy slightly above the first Lagrange point.

In fact, below the first Lagrange point, a regularized level set of the planar restricted 3-body problem is contactomorphic to $(\mathbb{R}P^3, \alpha_0) \cong (S^3/\mathbb{Z}_2, \alpha_0)$. Above the first Lagrange point it is contactomorphic to $(\mathbb{R}P^3, \alpha_0)\#(\mathbb{R}P^3, \alpha_0)$.

The proof is elementary: we find Liouville vector fields transverse to level sets of the form

$$q\partial_q$$

This is peculiar, since one usually has $p\partial_p$ for mechanical systems.

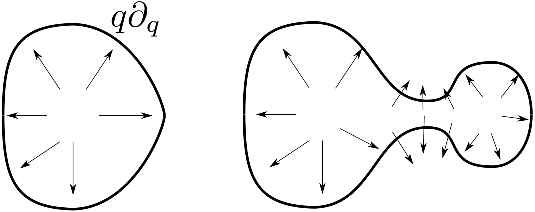
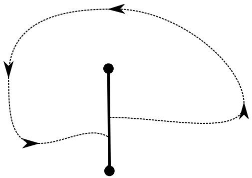


Figure: Liouville vector fields to "contact" the moon

Beyond the first Lagrange points we interpolate the Liouville vector fields for the Earth and the Moon.

Finite energy foliations

Let M be a 3-manifold with a non-vanishing vector field X. In order to understand the flow of X we want to discretize the flow by a **surface of section** S: this surface should be transverse to the flow and an orbit intersecting S must return.



We then get the **return map**

$$\phi : S \longrightarrow S.$$

With the return map we can understand the dynamics more easily. For instance, fixed points of ϕ and its iterates are periodic orbits of the original flow.

Hofer, Wysocki and Zehnder found an abstract means to construct surfaces of section using holomorphic curve methods (solutions to a certain PDE). In order to control the behavior of these holomorphic curves, the contact condition is of great importance.

Let J be a compatible almost complex structure for a symplectic manifold $(W^{2n}, \omega = d\lambda)$ with contact end (M, λ).
This means that $J : TW \to TW$ satisfies $J^2 = -\mathrm{Id}$, $\omega(\ldots, J\ldots)$ is inner product on TW.

Consider the non-linear Cauchy-Riemann equation for a map $u : \Sigma \to W$, where (Σ, j) is a punctured Riemann surface,

$$du + J \circ du \circ j = 0.$$

Near the punctures we impose asymptotic boundary conditions: u should converge to a periodic Reeb orbit. Solutions are called **holomorphic curves**. Alternatively, one can impose a finite energy condition.

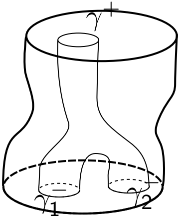


Figure: Finite energy J-curve in a symplectic manifold

We collect the solutions of this PDE up to automorphism of Σ in a moduli space: $\mathcal{M}^{W^{2n}}(\gamma_1^+, \ldots, \gamma_k^+; \gamma_1^-, \ldots, \gamma_l^-)$. For regular J this is an orbifold of dimension
$\sum_j \mu_{CZ}(\gamma_j^+) - \sum_j \mu_{CZ}(\gamma_j^-) + (n-3)(2-k-l)$.

Here μ_{CZ} is the Conley-Zehnder index, a kind of winding number of the linearized flow near an orbit.

Remark

For ST^*M the Reeb flow correponds to the geodesic flow. Furthermore, the Conley-Zehnder index of a periodic is equal to the Morse-index of a geodesic, i.e. the index of the energy functional

$$E(\gamma) = \int_a^b g(\dot{\gamma}, \dot{\gamma}) dt$$

Let (M, λ, J) be a contact manifold with compatible complex structure J. Extend J to the symplectization $\mathbb{R} \times M$: write $\tilde{J}$

Definition

A **finite energy foliation** is a smooth foliation $\mathcal{F}$ for $\mathbb{R} \times M$ such that

- every leaf F is the projection of an embedded $\tilde{J}$-holomorphic curve of finite energy.
- the foliation respects $\mathbb{R}$-translations in the symplectization direction.

Furthermore, there is a uniform energy bound for the holomorphic curves.

Such an energy bound implies that the curve is asymptotic to periodic Reeb orbits.

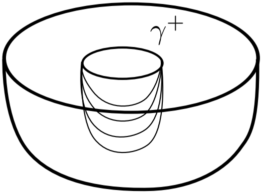


Figure: Finite energy foliation asymptotic to γ^+

With the above dimension formula, we see that such a foliation requires Reeb orbits with sufficiently high Conley-Zehnder index.
Best case: all indices of contractible Reeb orbits are at least 3: this is called **dynamically convex**. We obtain a finite energy foliation consisting of planes.
In particular, we get a disk-like surface of section.

Lemma

$(S^3, \alpha_0 = \frac{i}{2}\sum_j (z_j d\bar{z}_j - \bar{z}_j dz_j))$ carries a finite energy foliation.

The symplectization of S^3 satisfies $(\mathbb{R} \times S^3, de^t\alpha_0) \cong (\mathbb{C}^2 - \{0\}, \omega_0)$ and the complex structure i on $\mathbb{C}^2$ is adjusted to the symplectization.
There is a finite energy foliation consisting of planes: use the family parametrized by $a \in \mathbb{C}$

$$\begin{aligned} u_a : \mathbb{C} &\longrightarrow \mathbb{C}^2 - \{0\} \\ z &\longmapsto (z, a). \end{aligned}$$

For $a \neq 0$ this is a finite energy plane projecting down to a disk with boundary $(e^{i\phi}, 0)$. $a = 0$ corresponds to the bounding orbit.

Above the Lagrange point it is known that index 2-orbits appear: there is no longer a global surface of section.
A similar phenomenon already happens in S^3:

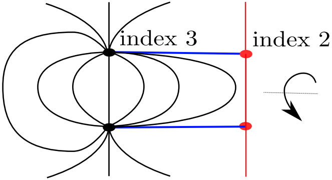


Figure: Breaking of a finite energy foliation in $S^3 \# S^3$

The blue segments indicate a rigid cylinder and the red segments two rigid planes.

How to obtain dynamical convexity below the Lagrange point?

- via Finsler geometry
- via direct computation (only for mass ratio $\mu = 0$)
- via convexity of the Levi-Civita embedding

But none of these methods work all the time.

Definition

Let M be a smooth manifold. A Finsler structure on M consists of $F : TM \to \mathbb{R}_{\geq 0}$ such that

- $F : TM - s_0 \to \mathbb{R}_{>0}$ is smooth.
- $F(x, \lambda v) = \lambda F(x, v)$ for $\lambda > 0$.
- $g_{ij} := \left(\frac{1}{2}F^2\right)_{y_i, y_j}$ is positive definite.

Note that g_{ij} defines a family of metrics in each tangent space $T_x M$ (as a function of the fiber coordinate).
Many notions in Riemannian geometry have analogues in Finsler geometry: curvature, geodesics

Proposition

The unit (co)-tangent bundle of a Finsler manifold carries a natural contact form. Finsler geodesics correspond to periodic Reeb orbits of this form.

Standard examples are Riemannian norms $\sqrt{g(\ldots,\ldots)}$, where g_{ij} just returns the Riemannian metric.
Given a Riemannian metric g and a small 1-form α, we can define a Randers metric,

$$F(x,y) := \sqrt{g_x(y,y)} + \alpha_x(y).$$

Theorem (Cieliebak, Frauenfelder, vK)

The rotating Kepler-problem ($\mu = 0$-case) is Finsler for all energies below the first Lagrange point.

If we consider the level set $H_K = -c$, then the Finsler structure has the form

$$F(p,q) = \frac{1}{4}(|p|^2 + 2c)|q| \left(1 + \sqrt{1 + \frac{16(q_1 p_2 - q_2 p_1)}{|q|(|p| + 2c)^2}}. \right)$$

here p is the base point, and q the vector (again in stereographic projection).

Theorem (Harris, Paternain)

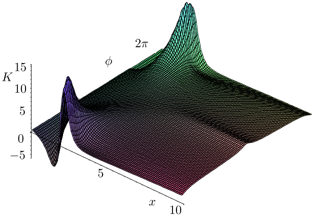
Suppose F is a Finsler metric on S^2 such that the curvature $K \geq \delta > 0$. If the minimal length of closed geodesics is greater than $\pi/\sqrt{\delta}$, then $S_F T^* S^2$ is dynamically convex.

There are different (and stronger) versions, but the basic idea is that positive curvature is good for dynamically convexity.

Unfortunately, we have

Theorem (Cieliebak, Frauenfelder, vK)

The flag-curvature of the Finsler metric associated with the rotating Kepler problem becomes negative.



On the other hand, we might be able to use this as a tool to find hyperbolic orbits.

Fortunately, one can compute the Conley-Zehnder/Maslov indices in the rotating Kepler problem more directly: The upshot is the following.

Theorem (Albers, Fish, Frauenfelder, vK)

The rotating Kepler problem is dynamically convex for $H = E < -3/2$.

Corollary

For μ *sufficiently small, there exists* $c(\mu) \geq 0$ *such that the restricted* 3*-body problem is dynamically convex for* $H = E < H(L_1) - c(\mu)$.

Here $c(\mu) \to 0$ as $\mu \to 0$.

Below the first Lagrange point a regularized component of the 3-body problem is $\mathbb{R}P^3$: we lift to its double cover S^3 by the Levi-Civita map

For the Kepler problem we have:

$$H(q,p)=\frac{1}{2}|p|^2-\frac{1}{|q|}.$$

Using the Levi-Civita map

$$\begin{aligned} q &= 2z^2 \\ p &= \frac{w}{\bar{z}} \end{aligned}$$

we can describe the dynamics at energy level $H=-c$ by the Hamiltonian

$$\tilde{K}=\frac{1}{2}\frac{|w|^2}{|z|^2}-\frac{1}{2|z|^2}+c.$$

This leads to the regularized Hamiltonian,

$$K = |w|^2 + 2c|z|^2 - 1.$$

Note that the level set $K = 0$ bounds a convex set.
This is also the case in general.

Theorem (Albers, Fish, Frauenfelder, Hofer, vK)

For all $\mu \in]0,1[$ there exists $c(\mu) > 0$ such that regularized level sets $H = E < H(L_1) - c(\mu)$ are convex. Furthermore, $c(\mu) \to 0$ as $\mu \to 1$.

Corollary

Such regularized level sets are dynamically convex.

Thank you

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