Real Analytic Metrics on S^2 with Total Absence of Finite Blocking

Lihuei Liu

March 7, 2012

Security

Definition

Let M be a Riemannian manifold. The pair $(x, y) \in M \times M$ is called *secure* if there exists a finite set $P \subset M \setminus \{x, y\}$ such that every geodesic from x to y passes through a point of P. Such P is called a *finite blocking set*.

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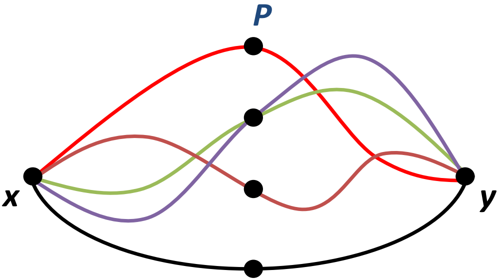

Finite Blocking Set

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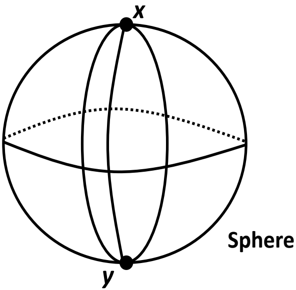

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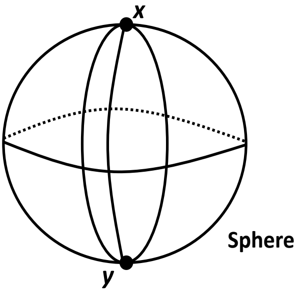

Definition

(M, g) is *totally insecure* if each $(x, y) \in M \times M$ is insecure.

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- *For genus* $= 1$, *there exists a* C^2*-open and* C^∞*-dense set,* $\mathcal{G}$, *of metrics, such that for any* $g \in \mathcal{G}$, (M, g) *is totally insecure.*

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(3) *Compact manifolds without conjugate points whose geodesic flows have positive topological entropy are totally insecure due to Burns and Gutkin(2008), independently, Lafont and Schmidt(2007).*

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What happens to the case of compact surfaces with genus zero?

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Theorem (M. Gerber and L. Liu, 2011)

There exists a totally insecure real analytic metric on S^2.

Construction of Real Analytic Metrics $\tilde{g}$

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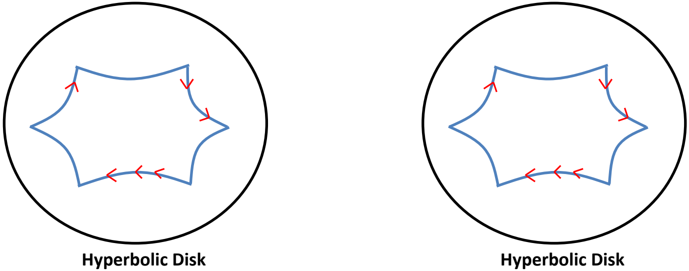

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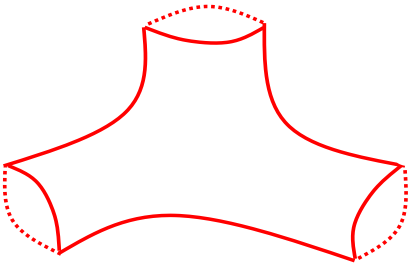

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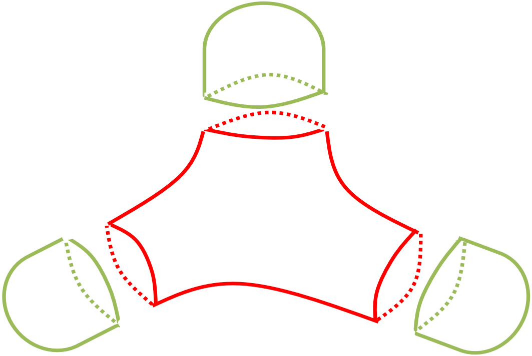

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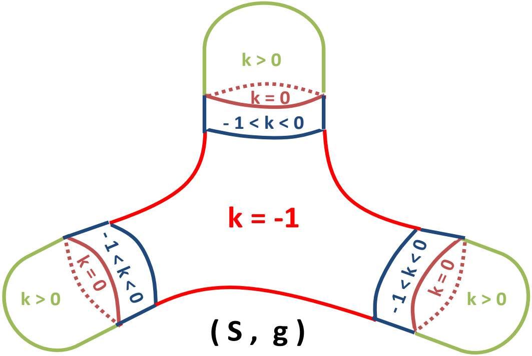
- A *cap* is defined to be a closed two-dimensional disk with nonnegative curvature such that the boundary is the trace of a real analytic geodesic.
- We say that a cap has *monotone curvature* if it is radially symmetric and its curvature is a nondecreasing function of distance from the boundary.

Construction of Real Analytic Metrics $\tilde{g}$

We obtain a surface S with C^∞ Riemannian metric g.



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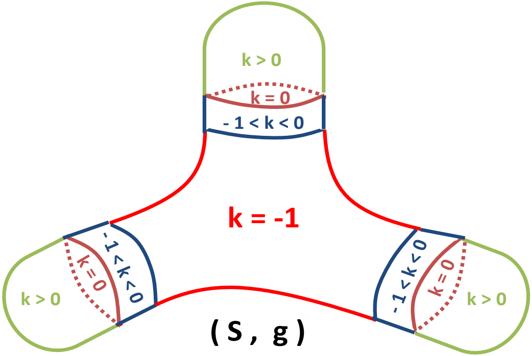

The geodesic flow for (S, g) is ergodic by Burns and Gerber(1989).

Construction of Real Analytic Metrics $\tilde{g}$

The compact surface (S, g) we constructed satifies the following condition: If s is the signed distance from the boundary of a cap $\mathcal{C} = \mathcal{C}_i$, for $i \in \{1, 2, 3\}$, with $s > 0$ in the interior of $\mathcal{C}$, then the curvature with respect to g is s for points in a neighborhood of $\partial\mathcal{C}$.

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Theorem (Cartan,1957)

Let S be a compact surface with a real analytic differentiable structure, and let g be a C^∞ Riemannian metric on S. Suppose that Γ is a union of disjoint closed real analytic curves on S and $\mathcal{U}$ is a neighborhood of Γ on which g is real analytic. Then for any $k \in \mathbb{N}$ there exists a real analytic metric $\tilde{g}$ on S such that g and $\tilde{g}$ agree up to order k on Γ. Moreover, $\tilde{g}$ can be taken arbitrarily to g in the C^∞ topology.

Construction of Real Analytic Metrics $\tilde{g}$

c. Applying Cartan's theorem to the surface (S, g), we obtain a real analytic metric $\tilde{g}$ on S such that $\tilde{g}$ and g agree up to second order on $\partial\mathcal{C}$ and $\tilde{g}$ is close to g in the C^∞ topology.

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Theorem (Burns and Gerber,1989)

Let (S, g) be the compact surface we constructed previously. If h is a C^3 Riemannian metric on S satisfying

- *h and g agree to second order on $\partial\mathcal{C}_i$, $i = 1, 2, 3$, and*
- *h is sufficiently C^3 close to g everywhere,*

then the geodesic flow for h is ergodic with respect to Liouville measure (which is positive on open sets). Moreover the family of metrics $\{h\}$ include real analytic metrics.

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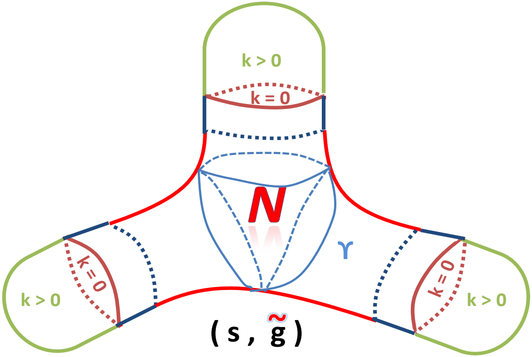
d. Therefore the geodesic flow for $(S, \tilde{g})$ is ergodic.

Construction of a closed geodesic γ on $(S, \tilde{g})$

Construction of a closed geodesic for $\tilde{g}$

Construction of a closed geodesic γ in the region $\mathcal{N}$ where the curvature in $\mathcal{N}$ is between $-1-\epsilon$ and $-1+\epsilon$.



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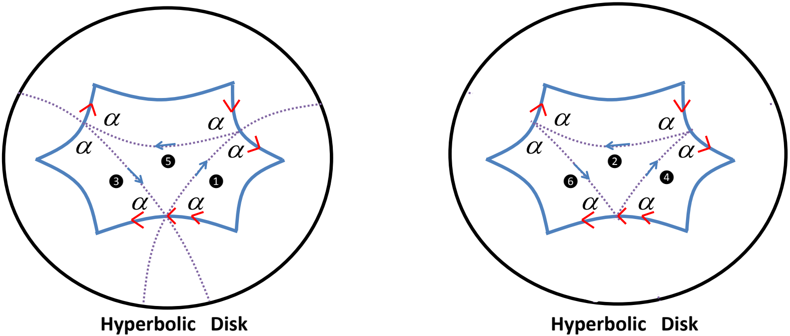

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b. The closed g-geodesic, say γ_g, we constructed lies in the region $k = -1$ on (S, g). Thus the orbit of the geodesic flow for g along γ_g is transversally hyperbolic.

Construction of a closed geodesic for $\tilde{g}$

b. The closed g-geodesic, say γ_g, we constructed lies in the region $k = -1$ on (S, g). Thus the orbit of the geodesic flow for g along γ_g is transversally hyperbolic.

c. By **b.** there exists a closed $\tilde{g}$-geodesic γ with self-intersection points in the region $\mathcal{N}$ where $\text{dist}(\mathcal{N}, \partial\mathcal{C}_i) > \delta$, $i = 1, 2, 3$ for some $\delta > 0$.

Existence of asymptotic geodesics to γ on $(S, \tilde{g})$

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For each cap $\mathcal{C}_i$, $i = 1, 2, 3$, we choose closed disks $\mathcal{D}_i$ and $\mathcal{E}_i$ in S that are radially symmetric about the center of $\mathcal{C}_i$ for the metric g such that $\mathcal{C}_i \subset \text{int}\mathcal{D}_i$ and $\mathcal{D}_i \subset \text{int}\mathcal{E}_i$. We require the disks $\mathcal{E}_i$ satisfy $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i \neq j$ and the trace of $\gamma \subset S \setminus (\cup_{i=1}^{3} \mathcal{E}_i)$.

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- If x is not the center of $\mathcal{D}$ in the radially symmetric g-metric, then the unit vector v_0 in the $\tilde{g}$-metric that is a positive multiple of $\sigma'(0)$ is called a *radial vector* at x.
- If x is the center of $\mathcal{D}$, then any $v_0 \in T_x^{1,\tilde{g}} S$ is called a radial vector at x.

Existence of asymptotic geodesics to γ

Lemma

1. *For all $x \in S$, there exist v_+ and $v_- \in T^1_x S$ such that $\gamma_{v_+}(t)$ and $\gamma_{v_-}(t)$ are asymptotic to γ and $-\gamma$, respectively, as $t \to \infty$. Moreover, if $x \in \mathcal{D}$, then v_+ and v_- can be chosen to be approximately radial vectors.*

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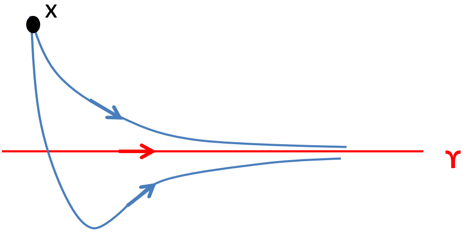

Geodesics asymptotic to ϒ

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2. *Given $x, y \in S$, $\varepsilon > 0$, there exist $T = T(\varepsilon)$ and an infinite family of distinct unit speed geodesics*

$$\gamma_n : [0, L_n] \longrightarrow S, \ \textit{with } L_n \longrightarrow \infty$$

from x to y such that

$$0 < \mathit{dist}(\gamma_n|_{[T, L_n - T]}, \gamma) < \varepsilon$$

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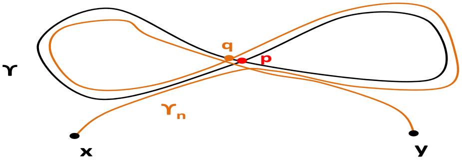

Lemma

3. *If* $T > 0, (x, z) \in (S, \tilde{g}) \times (S, \tilde{g})$, *and* $\gamma_n : [0, T] \to (S, \tilde{g}), n = 1, 2, \cdots$ *is an infinite sequence of distinct unit speed geodesics with* $\gamma_n(0) = x$, *then there are at most finitely many positive integers* n *such that* $z \in \gamma_n((0, T])$.

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Example

S^2 with round metric and the ellipsoid of revolution $\frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1$ in $\mathbb{R}^3$ are real analytic surfaces that fail to satisfy the conclusion of Lemma 3.

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Theorem (Łojasiewicz,1964)

Let $\mathbb{M}$ be a connected real analytic surface, $\mathbb{K}$ is a compact subset of $\mathbb{M}$, and $f : \mathbb{M} \longrightarrow \mathbb{R}$ is a real analytic function that does not vanish identically on $\mathbb{M}$, then there exist finitely many points $p_1, p_2, ..., p_n$ and finitely many real analytic curves $\alpha_1, \alpha_2, ..., \alpha_m$ such that

$$\{z \in \mathbb{K} : f(z) = 0\} = \mathbb{K} \cap \{\{p_1, p_2, ..., p_n\} \cup \alpha_1 \cup \alpha_2 \cup ... \cup \alpha_m\}.$$

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where $v \in T_x S$ and $\|v\| \leq T$.

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4. There exists an open disk $\mathcal{M}$ about $t_0 v_0$ in $T_x S$ such that $f|_{\mathcal{M}}$ is real analytic. Let $\mathcal{K}$ be a closed disk about $t_0 v_0$ that is contained in $\mathcal{M}$.

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5. Since $f(v) = 0$ for infinitely many $v's$ in $\mathcal{K}$, by theorem there exists a curve $\alpha(s)$, $-\delta < s < \delta$ in $T_x S$ such that $f(\alpha(s)) = 0$, $-\delta < s < \delta$.

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6. For each s, $\exp_x(t\alpha(s)), 0 \leq t \leq 1$ is a geodesic from x to z.
7. By the first variation formula for arc length, $\frac{d}{ds}\|\alpha(s)\| \equiv 0$. That implies there exists $L > 0$ such that $\|\alpha(s)\| \equiv L$.

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6. For each s, $\exp_x(t\alpha(s)), 0 \leq t \leq 1$ is a geodesic from x to z.
7. By the first variation formula for arc length, $\frac{d}{ds}\|\alpha(s)\| \equiv 0$. That implies there exists $L > 0$ such that $\|\alpha(s)\| \equiv L$.
8. f vanishes on an arc of the circle $\{v \in T_x S : \|v\| = L\}$. It follows that f vanishes on the circle $\{v \in T_x S : \|v\| = L\}$.

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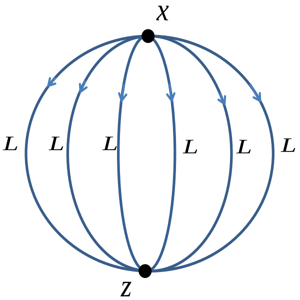

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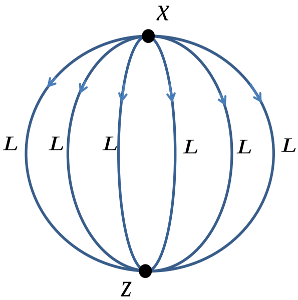

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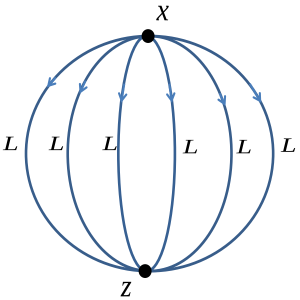

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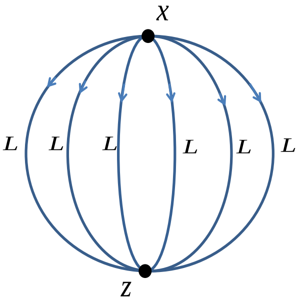

10. Same argument shows 9 with x and z switched.
11. It follows that every geodesic starting at x hits z infinitely many times.
12. By Lemma 1, there exists a geodesic starting at x that becomes asymptotic to γ, which contradicts 11.

Proof of Theorem

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Fact: A pair of points (x, y) is insecure if there exists an infinite family of geodesics $\{\gamma_n\}_{n=1}^{\infty}$ from x to y such that no three of $\{\gamma_n\}_{n=1}^{\infty}$ are concurrent at any point except x and y.

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4. Let

$$\Sigma = \{p : p \text{ is a self intersection point of } \gamma\}$$

$$\alpha_1 = \min\{\text{dist}(z_i, \gamma) : z_i \notin \gamma, 1 \leq i \leq m\}$$

$$\alpha_2 = \min\{\text{dist}(z_i, \Sigma) : z_i \in \gamma,\ z_i \notin \Sigma, 1 \leq i \leq m\}$$

$$\alpha = \min\{\alpha_1, \alpha_2\} > 0.$$

Proof of Theorem

5. We apply Lemma 2 with $\varepsilon < \alpha$ to obtain an infinite family of distinct unit speed geodesics

$$\sigma_n : [0, L_n] \longrightarrow S$$

from x to y such that

$$0 < \mathrm{dist}(\sigma_n|_{[T, L_n - T]}, \gamma) < \varepsilon.$$

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- Let

$$\Omega = \{q : q \in \sigma_n|_{[T, L_n - T]} \cap \text{the image of } \gamma, n = 1, 2, ...\}.$$

We have

$$0 < \mathrm{dist}(\Omega, \Sigma) < C\varepsilon$$

where $C > 0$ is independent of n.

Proof of Theorem

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6. Thus by induction we obtain an infinite family of geodesics from x to y such that no three are concurrent except at x and y.

Construction of Cones and Line Fields

Construction of Cones and Line Fields

Recall that a Riemannian metric $<,>$ on S induces a Riemannian metric on TS:

$$\ll \xi, \eta \gg = < \xi_H, \eta_H > + < \xi_V, \eta_V >$$

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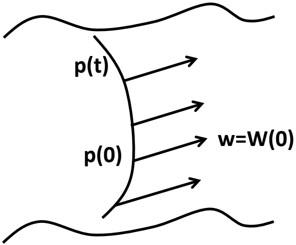
Let $\xi \in T_w TS$ and $\sigma(t) = (p(t), W(t))$ be a curve in TS that is tangent to ξ at w.

Construction of Cones and Line Fields



$W(t) \in T_{p(t)}S$

$\xi \leftrightarrow (\xi_H, \xi_V)$

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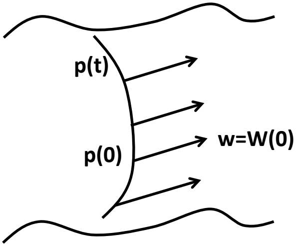

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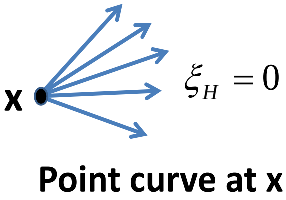
We identify ξ with (ξ_H, ξ_V).

Construction of Cones and Line Fields

Example



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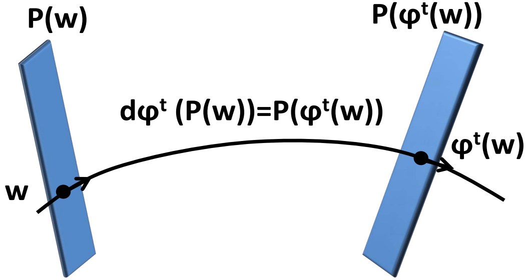
- If φ^t is the geodesic flow on T^1S, then the distribution P is invariant under $d\varphi^t$.

Construction of Cones and Line Fields



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Construction of Cones and Line Fields

Let $\xi \in T_w T^1 S$ and $J(t)$ be a perpendicular Jacobi field along the geodesic $\gamma_w(t)$ with $J(0) = \xi_H$ and $J'(0) = \xi_V$.

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Then $J(t) = j(t)N(t)$ where $N(t)$ is a normal field along $\gamma_w(t)$ and $j(t)$ satisfies the scalar Jacobi equation:

$$j''(t) + K(\gamma_w(t))j(t) = 0$$

where K is the Gaussian curvature of the surface S.

Construction of Cones and Line Fields

From the usual procedure for constructing horocycles (limit curves of geodesic circles) in regions of nonpositive curvature, we know that for each $x \in \mathcal{D}_i \setminus \mathcal{C}_i$ there exist exactly two vectors $v_{x,j} \in T^1_x S$, $j = 1, 2$ (corresponding to two possible orientations on $\partial\mathcal{C}_i$) such that

- $\gamma_{v_{x,j}}(t) \in \mathrm{int}(\mathcal{D}_i \setminus \mathcal{C}_i)$ for all $t < 0$,
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Consider $x \in \text{int}(\mathcal{D}_i \setminus \mathcal{C}_i)$ and $x \in \mathcal{C}_i$.

Construction of Cones and Line Fields

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- There exist $a < c < d < 0$ such that $\gamma_w(t) \in \mathrm{int}(\mathcal{D}_i \setminus \mathcal{C}_i)$ for $t \in (a, c) \cup (d, 0]$, $\gamma_w(t) \in \mathrm{int}\mathcal{C}_i$ for $t \in (c, d)$ and $\gamma_w(a) \in \partial\mathcal{D}_i$

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Case 2 $x \in \mathcal{C}_i$
If $w \in T_x^1 S \setminus T^1(\partial\mathcal{C}_i)$, then $\gamma_w(t)$ exits $\mathcal{D}_i$ in negative time. So we have:

- There exist $a < c \leq 0$ such that $\gamma_w(t) \in \text{int}\mathcal{C}_i$ for $c < t < 0$, $\gamma_w(t) \in \text{int}(\mathcal{D}_i \setminus \mathcal{C}_i)$ for $t \in (a, c)$, and $\gamma_w(a) \in \partial\mathcal{D}_i$.

Construction of Cones and Line Fields

Definition

For $w \in T^1_x S$, we define cones $\mathcal{K}^+_w$, $\mathcal{K}^-_w \subset P(w)$ by

$$\mathcal{K}^+_w = \{\xi \in P(w) : \langle \xi_H, \xi_V \rangle \geq 0\}$$

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Definition

a. If $x \in \text{int}\mathcal{D}$, for $\mathcal{D} = \mathcal{D}_i$, $i \in \{1, 2, 3\}$ and $w \in T^1_x S$ is such that there exists $a < 0$ with $\gamma_w(t) \in \text{int}\mathcal{D}$ for $t \in (a, 0]$ and $\gamma_w(a) \in \partial\mathcal{D}$, we define the *unstable cone* $\mathcal{K}^u_w$ by,

$$\mathcal{K}^u_w = d\varphi^{-a}\mathcal{K}^+_{\gamma'_w(a)}.$$

Construction of Cones and Line Fields

b. For all other $w \in T^1S$, we define

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Similarly we define the *stable cone* $\mathcal{K}^s$ as follows.

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b. For all other $w \in T^1 S$, we define

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Construction of Cones and Line Fields

The unstable(stable) cones are invariant for $d\varphi^t$, $t \geq 0 (t \leq 0)$. That is,

$$d\varphi^t \mathcal{K}^u_w \subset \mathcal{K}^u_{\varphi^t w}, \text{ for } t \geq 0,$$
$$d\varphi^t \mathcal{K}^s_w \subset \mathcal{K}^s_{\varphi^t w}, \text{ for } t \leq 0.$$

Moreover, if the basepoint of $\varphi^{\bar{t}} w$, for some $\bar{t} \in (0, t)$, lies outside $\mathcal{D}$, then

$$d\varphi^t \mathcal{K}^u_w \subset \text{int}\mathcal{K}^u_{\varphi^t w}, \text{ for } t > 0,$$
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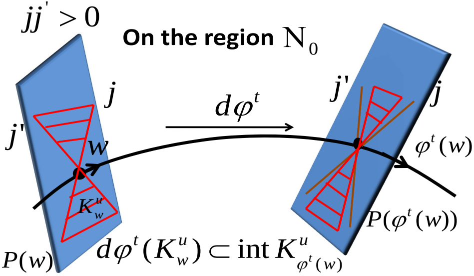
Construction of Cones and Line Fields

Let

$$\mathcal{N}_0 = S \setminus \cup_{i=1}^{3} \mathcal{D}_i.$$



$$(jj')' = (j')^2 + jj'' = (j')^2 - Kj^2 > 0 \text{ if } K < 0.$$

Construction of Cones and Line Fields

Let $w \in T^1S$. Note that $\{d\varphi^t(\mathcal{K}^u_{\varphi^{-t}(w)})\}_{t>0}$ is a nested sequence of cones. We define

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Let $w \in T^1S$. Note that $\{d\varphi^t(\mathcal{K}^u_{\varphi^{-t}(w)})\}_{t>0}$ is a nested sequence of cones. We define

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Similarly we can define the stable cone

$$\mathcal{K}^s_w = \overline{P(w) \setminus \mathcal{K}^u_w}$$

and the stable line field

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Define the Lyapunov function by

$$Q(\xi) = sgn(uv)\sqrt{|uv|}.$$

Then

$$Q(\xi) \leq \|\xi\| = \sqrt{u^2 + v^2}.$$

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Outline of Proofs of Parts of Lemma 1 and Lemma 2

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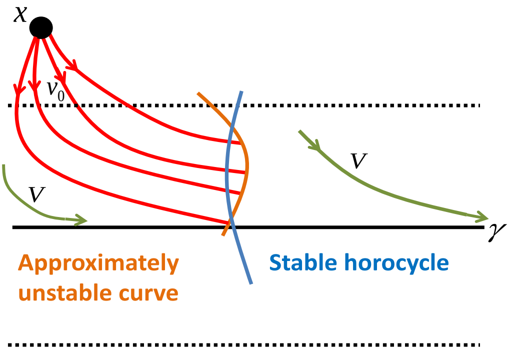

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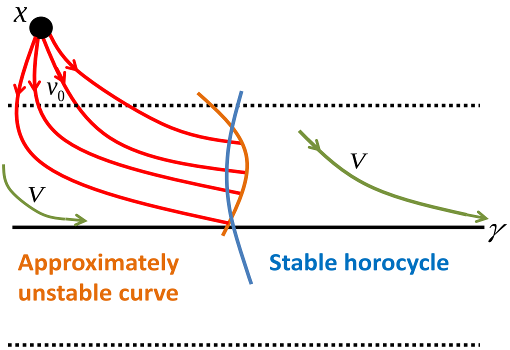

2. Take a wedge around (x, v_0), $\mathcal{W}(s) = (x, v(s))$, $\|v(s)\| \equiv 1$, $-\varepsilon \leq s \leq \varepsilon$. Consider the flow $\varphi^t(\mathcal{W}(s))$.

Outline of Proofs of Lemma 1

3. If $x \in \mathcal{D}$ then v_0 is chosen to be an approximately radial vector so that $\mathcal{W}'(s)$ is in the unstable cone $\mathcal{K}^u_{\mathcal{W}(s)}$.

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4. Let $0 < t_0 < \tilde{t}_0 < \tilde{t}_1 < t_1$. Consider

$$\mathcal{A} = \{\varphi^t(\mathcal{W}(s)) : -\varepsilon \le s \le \varepsilon, t_0 \le t \le t_1\}$$

and

$$\tilde{\mathcal{A}} = \delta\text{−neighborhood of } \{\varphi^t(\mathcal{W}(s)) : -\frac{\varepsilon}{2} \le s \le \frac{\varepsilon}{2}, \tilde{t}_0 \le t \le \tilde{t}_1\}.$$

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5. For δ sufficiently small, every vector $w \in \tilde{\mathcal{A}}$ can be joined to a vector in $\mathcal{A}$ by a stable curve.

Outline of Proofs of Lemma 1

6. Let $0 < \varepsilon_0$. Let

$$\mathcal{U}_0 = \{v \in T_p^1 S, p \in \mathcal{N}_0 : \sphericalangle(v, V(p)) < \varepsilon_0, \langle v, \frac{\partial}{\partial \tau_1} \rangle < \varepsilon_0$$

$$\text{and } \langle v, \frac{\partial}{\partial \tau_2} \rangle > \langle V(p), \frac{\partial}{\partial \tau_2} \rangle \}$$

where (τ_1, τ_2) is a Fermi coordinate on $\mathcal{N}_0$. Let $\mathcal{U}_1$ be an open set such that $\overline{\mathcal{U}_1} \subset \mathcal{U}_0$.

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8. Let $\sigma(s)$ be a stable curve from w_0 to $v_1 \in \mathcal{A}$.
9. We can show that there exists arbitrarily large t such that $\varphi^t(v_1) \in \mathcal{U}_0$.

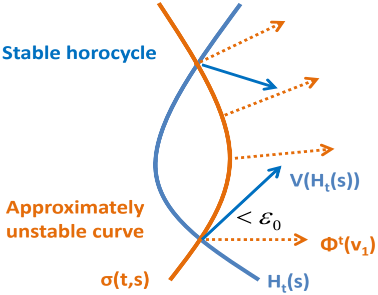
Outline of Proofs of Lemma 1

10. Let

$$\mathcal{H}_t = \text{the horocycle through } \varphi^t(v_1)$$

and

$$\varphi^t(\mathcal{W}(s)) = (\sigma(t,s), v(t,s)), \quad -\varepsilon \le s \le \varepsilon.$$



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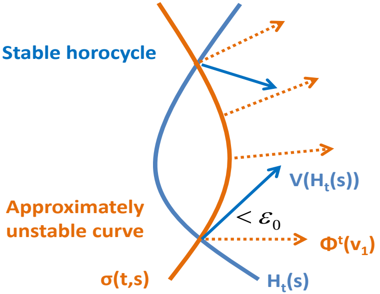

If ε_0 is sufficiently small, then there exists $\tilde{t} > 0$ such that there exists second intersection point of the curves $\mathcal{H}_{\tilde{t}}(s)$ and $\sigma(\tilde{t}, s)$ within $\mathcal{N}_0$ and on the same side of γ.

Outline of Proofs of Lemma 1

11. Applying the intermediate value theorem to the angle between $V(\mathcal{H}_t(s))$ and $v(t,s)$ along $\mathcal{H}_t(s)$ there exists $\tilde{s} \in (s_0, s_1) \subset (-\varepsilon, \varepsilon)$ such that $v(\tilde{t}, \tilde{s}) = V(\sigma(\tilde{t}, \tilde{s}))$.

Outline of Proofs of Lemma 2

1. Let $x, y \in S$ and $\varepsilon > 0$. By Lemma 1, there exist v_x at x and v_y at y such that the geodesics γ_{v_x} and γ_{v_y} are asymptotic to the closed geodesic γ and $-\gamma$ respectively on $(S, \tilde{g})$.

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2. Take wedges $\mathcal{W}_x = (x, v_x(s))$, $-\varepsilon_x \leq s \leq \varepsilon_x$ and $\mathcal{W}_y = (y, v_y(s))$, $-\varepsilon_y \leq s \leq \varepsilon_y$ around (x, v_x) and (y, v_y) respectively.

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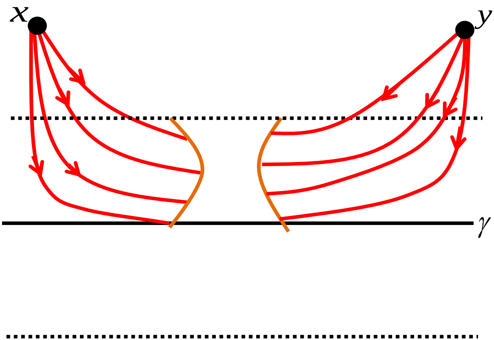

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4. Let $T > 0$ be such that

$$\text{dist}(\gamma_{v_x}(T, \infty), \gamma) < \frac{\varepsilon}{2}$$

$$\text{dist}(\gamma_{v_y}(T, \infty), \gamma) < \frac{\varepsilon}{2}$$

Outline of Proofs of Lemma 2

5. γ_{v_x} and γ_{v_y} intersect infinitely many times in $\mathcal{N}_{\frac{\varepsilon}{4}}$. Let $\gamma_{v_x}(t_n) = \gamma_{v_y}(s_n)$ be the n^{th} intersection point in $\mathcal{N}_{\frac{\varepsilon}{4}}$.

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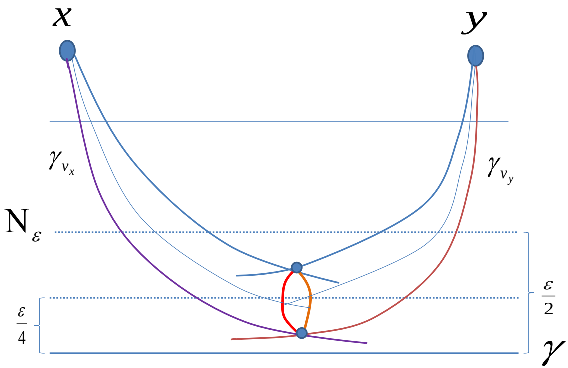

Outline of Proofs of Lemma 2

6. Let

$$\varphi^t(\mathcal{W}_x(s)) = (\sigma_x(t,s), v_x(t,s))$$

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There exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\sphericalangle(\frac{d}{ds}|_{s=0}\sigma_x(t_{n_k}, s), \frac{d}{ds}|_{s=0}\sigma_y(s_{n_k}, s)), \ \ k = 1, 2...$$

are small enough that the second intersection points, say

$$\sigma_x(t_{n_k}, \varepsilon_k(x)) = \sigma_y(s_{n_k}, \varepsilon_k(y))$$

$k = 1, 2....$, of the curves are within $\mathcal{N}_{\frac{\varepsilon}{2}}$.

Outline of Proofs of Lemma 2

- Note that the intersection points are on the same side of the geodesic γ by restricting $\mathcal{W}_x$ to $[0, \varepsilon_k(x)]$ or $[-\tilde{\varepsilon}_k(x), 0]$ and same thing is true for wedge $\mathcal{W}_y$.

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7. We know that the Lyapunov length of φ^t(approximately unstable curves) is a nondecreasing function of t. So we obtain $\sigma_x(t,s) \sim \gamma_{v_x}(t)$ on $[0, t_{n_k}]$ for each $s \in [0, \varepsilon_k(x)]$. The same is true for the case of y.

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8. Similar argument as 11 in proof of Lemma 1 leads to an infinite family of geodesics γ_{n_k}, $k = 1, 2, \ldots$ from x to y such that $\text{dist}(\gamma_{n_k}|_{[T, L_{n_k} - T]}, \gamma) < \varepsilon$, $k = 1, 2, \ldots$

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4. Well-known conjecture: (M, g) is secure $\implies$ (M, g) is flat?
   Special case: Show there does not exist a secure metric on S^2.