On a free boundary problem for a two-species weak competition system

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Lotka-Volterra competition-diffusion model

Let us start with the following Lotka-Volterra type competition-diffusion system for two species in a 1D habitat:

$$u_t = u_{xx} + u(1 - u - kv), \quad x \in \mathbb{R}, t \in \mathbb{R}, \tag{0.1}$$

$$v_t = Dv_{xx} + rv(1 - v - hu), \quad x \in \mathbb{R}, t \in \mathbb{R}, \tag{0.2}$$

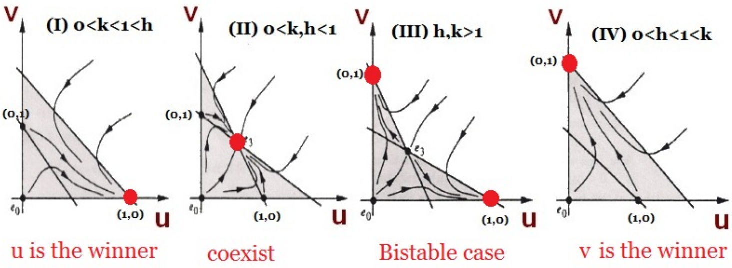
where all parameters are positive and

- $u(x,t), v(x,t)$: population densities of two competing species
- D: diffusion coefficient of v
- k, h: competition coefficients of species
- r: growth rate of species v

Spatially homogeneous case

$$\begin{aligned}
&\text{(I)} \quad 0<k<1<h \Longrightarrow (u,v)(t)\to(1,0) \text{ as } t\to\infty,\\
&\text{(II)} \quad 0<h,k<1 \Longrightarrow (u,v)(t)\to(u^*,v^*) \text{ as } t\to\infty,\\
&\text{(III)} \quad h,k>1 \Longrightarrow (1,0),(0,1) \text{ are local stable},\\
&\text{(IV)} \quad 0<h<1<k \Longrightarrow (u,v)(t)\to(0,1) \text{ as } t\to\infty.
\end{aligned}$$

We only consider the case $0<h,k<1$.



- Consider (0.1)-(0.2) with initial data $u_0(x), v_0(x) \geq 0$, which attain zero outside a bounded set:
  (A) $0 < k < 1 < h \Longrightarrow (u, v) \to (1, 0)$ as $t \to \infty$,
  (B) $0 < h, k < 1 \Longrightarrow (u, v) \to (u^*, v^*)$ as $t \to \infty$,
  (C) $h, k > 1 \Longrightarrow (1, 0), (0, 1)$ are local stable,
  (D) $0 < h < 1 < k \Longrightarrow (u, v) \to (0, 1)$ as $t \to \infty$.
- Consider traveling wave solutions (0.1)-(0.2):

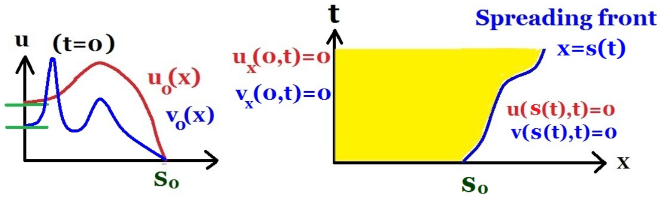
$$(u, v)(x, t) = (U, V)(\xi),$$

where $\xi = x - ct$ and c is the wave speed.

We consider the following problem **(P)** with $0 < h, k < 1$:

$$
\begin{aligned}
&u_t = u_{xx} + u(1 - u - kv),\ 0 < x < s(t),\ t > 0,\\
&v_t = Dv_{xx} + rv(1 - v - hu),\ 0 < x < s(t),\ t > 0,\\
&u_x(0,t) = v_x(0,t) = 0,\ u(s(t),t) = v(s(t),t) = 0,\ t > 0,\\
&s'(t) = -\mu[u_x(s(t),t) + \rho v_x(s(t),t)],\ t > 0, \quad (\mu, \rho > 0)
\end{aligned}
$$

with initial data:



Unknown: $(u(x,t), v(x,t), s(t))$

Free boundary in ecological models

- Mimura-Yamada-Yotsutani (1985,1986)

$$\begin{cases} u_t = d_1 u_{xx} + uf(u), \ 0 < x < h(t), \ t > 0, \\ v_t = d_2 v_{xx} + vg(v), \ h(t) < x < L, \ t > 0, \\ u(0,t) = M_1, \ v(L,t) = M_2, \ t > 0, \\ u(h(t),t) = v(h(t),t) = 0, \ t > 0, \\ h'(t) = -\mu_1 u_x(h(t),t) - \mu_2 v_x(h(t),t), \ t > 0, \\ h(0) = h_0, \\ u(x,0) = u_0(x), \ 0 < x < h_0, \ v(x,0) = v_0(x), \ h_0 < x < L. \end{cases}$$

- Du-Lin (2010):

$$\begin{cases} u_t = du_{xx} + u(a - bu), \ 0 < x < h(t), \ t > 0, \\ u_x(0,t) = 0, \ u(h(t),t) = 0, \ t > 0, \\ h'(t) = -\mu u_x(h(t),t), \ t > 0, \\ h(0) = h_0, \ u(x,0) = u_0(x), \ 0 < x < h_0, \end{cases}$$

Free boundary in biological models

- Chen and Friedman, SIAM J. Math. Anal. (2000,2003)
- Du and Guo, J. Diff. Eqns. (2011)
- Hilhorst, Mimura, and Schatzle, Nonlinear Anal. RWA (2003)
- Lin, Nonlinearity (2007)
- M. Mimura, Y. Yamada, S. Yotsutani, Hiroshima Math. J. (1987)
- .....

- The two species vanish eventually: $s_\infty < \infty$ and

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{C[0,s(t)]} = \lim_{t\to+\infty} \|v(\cdot,t)\|_{C[0,s(t)]} = 0$$

- The two species spread successfully: $s_\infty = +\infty$ and $\liminf_{t\to+\infty} u(x,t) > 0$ and $\liminf_{t\to+\infty} v(x,t) > 0$ locally uniformly in $x \in [0,+\infty)$.

What we try to understand is ...

Theorem (Existence and uniqueness)

(P) *admits a unique global (in time) solution* $(u, v, s) \in C^{2,1}(\Omega) \times C^{2,1}(\Omega) \times C^1([0, \infty))$, *where* $\Omega := \{(x, t) :\ 0 \le x \le s(t),\ \ t > 0\}$, *such that*

$$\begin{aligned} &0 < u(x,t) \le \max\{1, \|u_0\|_{L^\infty[0,s_0]}\} \quad \text{for } x \in [0, s(t)),\ \ t > 0,\\ &0 < v(x,t) \le \max\{1, \|v_0\|_{L^\infty[0,s_0]}\} \quad \text{for } x \in [0, s(t)),\ \ t > 0,\\ &0 < s'(t) \le \mu\Lambda \quad \text{for } t > 0, \end{aligned}$$

where Λ *dose not depend on* μ, h *and* k.

- Since $s'(t) > 0$, we can define $s_\infty := \lim_{t\to+\infty} s(t)$

Define two positive quantity:

- $s_* := \min\left\{\frac{\pi}{2}, \frac{\pi}{2}\sqrt{\frac{D}{r}}\right\}$
- Note that $0 < h, k < 1$,

$$s^* := \begin{cases} \dfrac{\pi}{2}\sqrt{\dfrac{D}{r}}\dfrac{1}{\sqrt{1-h-\frac{1}{rD}(\frac{\mu\Lambda}{2})^2}} & \text{if } D < r; \\ \dfrac{\pi}{2}\dfrac{1}{\sqrt{1-k-(\frac{\mu\Lambda}{2})^2}} & \text{if } D > r; \\ \min\left\{\dfrac{\pi}{2}\dfrac{1}{\sqrt{1-k-(\frac{\mu\Lambda}{2})^2}},\ \dfrac{\pi}{2}\dfrac{1}{\sqrt{1-h-\frac{1}{rD}(\frac{\mu\Lambda}{2})^2}}\right\} & \text{if } D = r. \end{cases}$$

- s^* depends on D, r, h, k, μ, ρ and the initial data
- $s_* < s^*$ if s^* exists

To make sure that s^* exists we make the following assumption

(A1) $0 < h, k < 1,\ r > 0,\ D > 0,\ \rho > 0,\ \mu \in (0, \mu^*)$ for some $\mu^* > 0$ which does not depend on the initial data.

- Recall $\Lambda > 0$ s.t. $s'(t) \leq \mu\Lambda$, where

$$\Lambda := 2M_1 \max\{1, \|u_0\|_{L^\infty}\} + 2\rho M_2 \max\{1, \|v_0\|_{L^\infty}\};$$
$$M_1 := \max\left\{\frac{4}{3}, \frac{-4}{3}\left(\min_{x\in[0,s_0]} u_0'(x)\right)\right\};$$
$$M_2 := \max\left\{\sqrt{\frac{r}{2D}}, \frac{4}{3}, \frac{-4}{3}\left(\min_{x\in[0,s_0]} v_0'(x)\right)\right\}.$$

- Under **(A1)**, s^* is well-defined if and only if

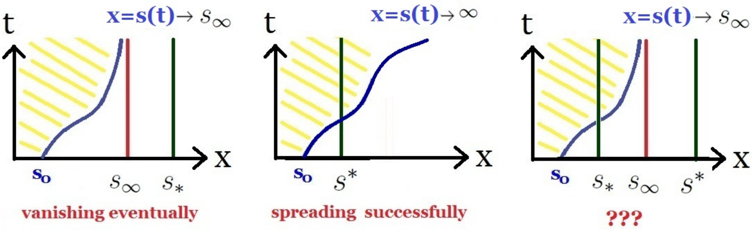
$$\Lambda < \begin{cases} 2\sqrt{rD(1-h)}\ /\mu, & \text{if } D < r; \\ 2\sqrt{(1-k)}\ /\mu, & \text{if } D > r; \\ \min\left\{2\sqrt{rD(1-h)}\ /\mu,\ 2\sqrt{(1-k)}\ /\mu\right\} & \text{if } D = r. \end{cases}$$

Theorem

Let (u, v, s) be a solution of **(P)**. *Then the followings hold.*

(i) *If $s_\infty \leq s_*$, then the two species vanish eventually.*

(ii) *If $s_\infty > s^*$, then the two species spread successfully.*



Theorem (A spreading-vanishing dichotomy)

Assume that **(A1)** *holds. Let* (u, v, s) *be a solution of* $(\mathbf{P})$ *with* $(D, h, k, r, \mu, \Lambda) \in A \cup B$, *where*

$$A := \left\{ D < r,\ \ h + \frac{1}{rD}\left(\frac{\mu\Lambda}{2}\right)^2 \leq 1 - \frac{D}{r} \right\}; \qquad (0.3)$$

$$B := \left\{ D > r,\ \ k + \left(\frac{\mu\Lambda}{2}\right)^2 \leq 1 - \frac{r}{D} \right\}. \qquad (0.4)$$

Then either $s_\infty \leq s_*$ *(and so the two species vanish eventually), or the two species spread successfully.*

s_* is the critical length !

Corollary

(i) *For given* (D, h, k, r, μ, ρ) *satisfying* **(A1)**, *the initial data such that*

$$\Lambda < \begin{cases} 2\sqrt{rD(1-h)}\ /\mu, & \text{if } D < r; \\ 2\sqrt{(1-k)}\ /\mu, & \text{if } D > r; \\ \min\left\{2\sqrt{rD(1-h)}\ /\mu,\ 2\sqrt{(1-k)}\,/\mu\right\} & \text{if } D = r. \end{cases}$$

and $s_0 \geq s^*$, *then the species* u *and* v *spread successfully.*

(ii) *Given the initial data* (u_0, v_0, s_0) *and* $(D, h, k, r, \mu, \Lambda) \in A \cup B$ *and* $s_0 \geq s_*$, *then the species* u *and* v *spread successfully.*

(iii) *If* $s_0 < s_*$, *then exists* $\beta(\mu, \rho, D, r, s_0) > 0$ *such that the species* u *and* v *vanish eventually as long as*

$$\max\{\,\|u_0\|_{L^\infty},\ \|v_0\|_{L^\infty}\,\} \leq \beta,$$

In the case of spreading success, we have the following more precise asymptotic behavior.

Theorem

Suppose that the two species spread successfully. Then

$$(u, v)(x, t) \to \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk} \right) \text{ as } t \to +\infty, \tag{0.5}$$

uniformly in any compact subset of $[0, +\infty)$.

The speed of the spreading front $x = s(t)$

Traveling wavefront solution:

$$(u, v)(x, t) = (U, V)(\xi), \quad \xi := x + ct$$

Look for (c, U, V) satisfying the following problem **(Q)**:

$$\begin{aligned} cU' &= U'' + U(1 - U - kV), \ \xi \in \mathbb{R}, \\ cV' &= DV'' + rV(1 - V - hU), \ \xi \in \mathbb{R}, \\ 0 &\le U, V \le 1, \ \xi \in \mathbb{R}, \end{aligned}$$

$$BC : (U, V)(-\infty) = \left(0, \frac{1-k}{1-hk}\right), \quad (U, V)(+\infty) = \left(0, \frac{1-h}{1-hk}\right).$$

Theorem (Tang-Fife (1980))

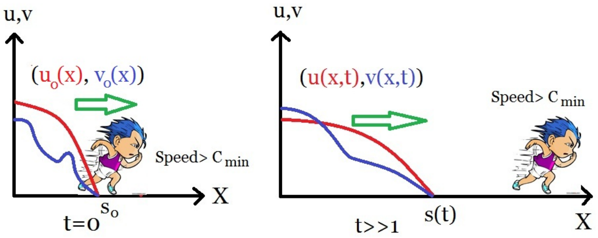
(Q) *has a solution if and only if* $c \ge c_{\min} := \max\{2, 2\sqrt{rD}\}$.

Here $c_{\min}$ is called the minimal speed of traveling wavefronts.

Theorem

Let (u, v, s) *be a solution of* $(\mathbf{P})$ *with* $s_\infty = +\infty$. *Then*

$$\limsup_{t\to+\infty} \frac{s(t)}{t} \le c_{\min} := \max\{2, 2\sqrt{rD}\}.$$



The standard comparison principle

Lemma

Let (u, v, s) *be a solution of* **(FBP)** *and* $(\bar{u}, \underline{v}) \in C^{2,1}(\mathcal{D}) \times C^{2,1}(\mathcal{D})$, *where* $\mathcal{D} := \{(x,t) : \ 0 \le x \le s(t), \ t > 0\}$, *satisfying the following:*

$$\begin{aligned}
&\bar{u}_t \ge \bar{u}_{xx} + \bar{u}(1 - \bar{u} - k\underline{v}) \text{ in } \mathcal{D},\\
&\underline{v}_t \le D\underline{v}_{xx} + r\underline{v}(1 - \underline{v} - h\bar{u}) \text{ in } \mathcal{D},\\
&\bar{u}_x(0,t) \le 0, \ \underline{v}_x(0,t) \ge 0, \ t > 0,\\
&\bar{u}(s(t),t) \ge 0, \ \underline{v}(s(t),t) \le 0, \ t > 0.
\end{aligned}$$

If $\bar{u}(x,0) \ge u_0(x)$, $\underline{v}(x,0) \le v_0(x)$ *for all* $x \in [0, s_0]$, *then* $\bar{u}(x,t) \ge u(x,t)$ *and* $\underline{v}(x,t) \le v(x,t)$ *for all* $x \in [0, s(t)]$, $t \ge 0$.

The comparison principle for free boundary

Lemma

Also assume that $(w_1, w_2, \sigma) \in C^{2,1}(\mathcal{D}) \times C^{2,1}(\mathcal{D}) \times C^1([0,\infty))$, *where* $\mathcal{D} := \{(x,t) : \ 0 \le x \le \sigma(t), \ t > 0\}$, *satisfying the following:*

$$\begin{aligned} &w_{1,t} \ge w_{1,xx} + w_1(1-w_1) \text{ in } \mathcal{D}, \\ &w_{2,t} \ge Dw_{2,xx} + rw_2(1-w_2) \text{ in } \mathcal{D}, \\ &w_{i,x}(0,t) \le 0, \ w_i(\sigma(t),t) = 0, \ t > 0, \ i = 1,2, \\ &\sigma'(t) \ge -\mu(1+\rho)w_{i,x}(\sigma(t),t), \ t > 0, \ i = 1,2. \end{aligned}$$

If $w_1(x,0) \ge u_0(x)$, $w_2(x,0) \ge v_0(x)$ *for all* $x \in [0, s_0]$ *and* $\sigma(0) \ge s_0$, *then* $\sigma(t) \ge s(t)$ *for all* $t \ge 0$, $w_1(x,t) \ge u(x,t) \ge 0$ *and* $w_2(x,t) \ge v(x,t) \ge 0$ *for all* $x \in [0, s(t)]$, $t \ge 0$.

Outline of proofs

Theorem

Let (u, v, s) be a solution of **(P)**. *Then the followings hold.*

(i) *If $s_\infty \le s_*$, then the two species vanish eventually.*

(ii) *If $s_\infty > s^*$, then the two species spread successfully.*

- Let $l \in [s_\infty, s_*]$, we can find $\bar{u}$ and $\bar{v}$ with

$$\lim_{t\to+\infty} \|\bar{u}(\cdot,t)\|_{C([0,l])} = \lim_{t\to+\infty} \|\bar{v}(\cdot,t)\|_{C([0,l])} = 0;$$

s.t. we can compare $(\bar{u}, 0)$ with (u, v) and $(0, \bar{v})$ with (u, v) respectively, over

$$\Omega := \{(x,t) \in \mathbb{R}^2 : \; 0 \le x \le s(t), \; t \ge 0\},$$

we obtain $0 \le u \le \bar{u}$ and $0 \le v \le \bar{v}$ in Ω.

Outline of proofs

Theorem

Let (u, v, s) be a solution of **(P)**. *Then the followings hold.*

(i) *If $s_\infty \leq s_*$, then the two species vanish eventually.*

(ii) *If $s_\infty > s^*$, then the two species spread successfully.*

- By the definition of s^*, we can construct suitable super-subsolution over

$$\Omega := \{(x, t) : 0 \leq x \leq s(t), t > T\},$$

by comparison, there exists $\kappa > 0$ s.t. $u_x(s(t), t) \leq -\kappa$ or $v_x(s(t), t) \leq -\kappa$ for all $t > T$.

Outline of proofs

Theorem (A spreading-vanishing dichotomy)

Assume that **(A1)** *holds. Let* (u, v, s) *be a solution of* $(\mathbf{P})$ *with* $(D, h, k, r, \mu, \Lambda) \in A \cup B$, *where*

$$A := \left\{ D < r,\ h + \frac{1}{rD}\left(\frac{\mu\Lambda}{2}\right)^2 \leq 1 - \frac{D}{r} \right\}; \tag{0.6}$$

$$B := \left\{ D > r,\ k + \left(\frac{\mu\Lambda}{2}\right)^2 \leq 1 - \frac{r}{D} \right\}. \tag{0.7}$$

Then either $s_\infty \leq s_*$ *(and so the two species vanish eventually), or the two species spread successfully.*

- When $D \neq r$, $s_\infty \notin \left(s_*,\ \max\left\{ \frac{\pi}{2}, \frac{\pi}{2}\sqrt{\frac{D}{r}} \right\} \right]$
- $s^* < \max\left\{ \frac{\pi}{2}, \frac{\pi}{2}\sqrt{\frac{D}{r}} \right\} \Longleftrightarrow$ The parameters and the initial data satisfies the assumption of this theorem

Outline of proofs

To prove the following theorem, we need two lemmas.

Theorem

Suppose that the two species spread successfully. Then

$$(u, v)(x, t) \to \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk} \right) \text{ as } t \to +\infty, \tag{0.8}$$

uniformly in any compact subset of $[0, +\infty)$.

Outline of proofs

Lemma

Assume that $0 < h, k < 1$.

(i) *Consider two sequences* $\{\bar{u}_n\}_{n\in\mathbb{N}}$ *and* $\{\underline{v}_n\}_{n\in\mathbb{N}}$ *defined as follows:*

$$(\bar{u}_1, \underline{v}_1) := (1, 1-h); \quad (\bar{u}_{n+1}, \underline{v}_{n+1}) := (1 - k\underline{v}_n, 1 - h(1 - k\underline{v}_n)).$$

Then $\bar{u}_n > \bar{u}_{n+1} > 0$ *and* $\underline{v}_n < \underline{v}_{n+1} < 1$ *for all* $n \in \mathbb{N}$. *Moreover,*

$$(\bar{u}_n, \underline{v}_n) \to \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk}\right) \text{ as } n \to +\infty.$$

(ii) *Consider two sequences* $\{\underline{u}_n\}_{n\in\mathbb{N}}$ *and* $\{\bar{v}_n\}_{n\in\mathbb{N}}$ *defined as follows:*

$$(\underline{u}_1, \bar{v}_1) := (1-k, 1); \quad (\underline{u}_{n+1}, \bar{v}_{n+1}) := (1 - k(1 - h\underline{u}_n), 1 - h\underline{u}_n).$$

Then $\underline{u}_n < \underline{u}_{n+1} < 1$ *and* $\bar{v}_n > \bar{v}_{n+1} > 0$ *for all* $n \in \mathbb{N}$. *Moreover,*

$$(\underline{u}_n, \bar{v}_n) \to \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk}\right) \text{ as } n \to +\infty.$$

Outline of proofs

Lemma

Let (u, v, s) *be a solution of* **(FBP)** *with* $s_\infty = +\infty$. *Then for each* $n \in \mathbb{N}$,

$$\underline{u}_n \leq \liminf_{t\to+\infty} u(x,t) \leq \limsup_{t\to+\infty} u(x,t) \leq \bar{u}_n,$$
$$\underline{v}_n \leq \liminf_{t\to+\infty} v(x,t) \leq \limsup_{t\to+\infty} v(x,t) \leq \bar{v}_n,$$

uniformly in any compact subset of $[0, +\infty)$.

Discussion

We study the system (0.1)- (0.2) via a free boundary problem:

- We provide some conditions such that the species spread successfully and vanish eventually respectively
- It can make the species die out eventually
- More realistic?
- The spreading speed can not be faster than minimal traveling wave speed

Thank you for your attention!