Fixed Point and Mean Convergence Theorems without Convexity and Related Topics

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Introduction

H: a Hilbert space,
$T : H \to H$: linear, contraction, i.e.,

$$\|Tx\| \le \|x\|, \quad \forall x \in H.$$

Then, for any $x \in H$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges strongly to $z \in F(T)$,
where $F(T)$ is the set of fixed points of T.

Introduction

H: a Hilbert space,
C: a nonempty closed convex subset of H,

$T : C \to C$: nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Theorem [Baillon, 1975]

H: a Hilbert space,
C: a nonempty closed convex subset of H,
$T : C \to C$, a nonexpansive mapping, $F(T) \neq \emptyset$.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z \in F(T)$.

For more general results,
see Takahashi [Proc. Amer. Math. Soc., 1981, 1986],
Lau and Takahashi [J. Func. Anal., 1996],
Shioji, Lau and Takahashi [J. Func. Anal., 1999]

On the other hand,
we know the following equilibrium problem
which includes, as special cases, optimization problems,
variational inequality problems, complementality problems,
fixed point problems, saddle point problems, and others.

H: a Hilbert space,
C: a closed convex subset of H,
$f : C \times C \to \mathbb{R}$, a bifunction

Find $z \in C$ such that

$$f(z, y) \geq 0, \quad \forall y \in C.$$

For solving the equilibrium problem, let us assume that

$f : C \times C \to \mathbb{R}$ satisfies (A1)–(A4):

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) $f(x, \cdot)$ is lower semicontinuous and convex for all $x \in C$;
(A4) $\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$ for all $x, y, z \in C$.

Lemma[Blum and Oettli]

For $r > 0$, $x \in H$,
there exists a unique $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Define $T_r : H \to C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}.$$

Then, T_r: firmly nonexpansive, i.e., for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle.$$

For $x, y \in H$,
$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2$.

$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$
$\iff 2\|Tx - Ty\|^2 \leq 2\langle Tx - Ty, x - y\rangle$
$\iff 2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty\|^2 - \|x - y - (Tx - Ty)\|^2$
$\iff \|Tx - Ty\|^2 \leq \|x - y\|^2 - \|x - y - (Tx - Ty)\|^2$
$\Longrightarrow \|Tx - Ty\|^2 \leq \|x - y\|^2$
$\iff \|Tx - Ty\| \leq \|x - y\|$.

$T : C \to H$: nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

For $x, y, z, w \in H$,
$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$.

$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$
$\iff 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2$
$\implies 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$.

$T : C \to H$: nonspreading if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2, \quad \forall x, y \in C.$$

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle$$
$$\iff 4\|Tx - Ty\|^2 \le 4\langle Tx - Ty, x - y\rangle$$
$$\iff 4\|Tx - Ty\|^2 \le \|Tx - Ty\|^2 + \|x - y\|^2 - \|x - y - (Tx - Ty)\|^2$$
$$\quad + \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2$$
$$\Longrightarrow 3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2.$$

$T : C \to H$: hybrid if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C.$$

Theorem [Takahashi and Yao, Taiwanese J. Math., 2010]

H: a Hilbert space,
C: a nonempty closed convex subset of H,
T: a mapping of C into itself such that $F(T)$ is nonempty.
Suppose that T satisfies one of the following:
(i) T is nonspreading;
(ii) T is hybrid;
(iii) $2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C$.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z \in F(T)$.

Can we unify Baillon's nonlinear ergodic theorem
and Takahashi and Yao's nonlinear ergodic theorem?

Definition [Kocourek, T. and Yao, Taiwanese J. Math., 2010]

H: a real Hilbert space,
C: a nonempty subset of H.
Then, a mapping $T : C \to H$ is called *generalized hybrid* if
there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$.

T is called an (α, β)-*generalized hybrid* mapping.

(1) $T: C \to H$: $(1,0)$-generalized hybrid if and only if
T is nonexpansive, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(2) $T: C \to H$ is $(2,1)$-generalized hybrid if and only if
T is nonspreading, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

(3) $T: C \to H$ is $(\frac{3}{2}, \frac{1}{2})$-hybrid if and only if
T is hybrid, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Theorem [Kocourek, T. and Yao, Taiwanese J. Math., 2010]

H: a Hilbert space
C be a nonempty closed convex subset of H.
$T : C \to C$, a generalized hybrid mapping with $F(T) \neq \emptyset$
P: the mertic projection of H onto $F(T)$.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in F(T)$, where
$p = \lim_{n\to\infty} PT^n x$.

(1) Can we prove nonlinear ergodic theorems without convexity in Hilbert spaces?

(2) Can we extend such theorems without convexity to Banach spaces?

Introduction

Our talk is organized as follows:

1. Nonlinear Ergodic Theorems without Convexity in Hilbert Spaces;

2. Strong Convergence Theorems without Convexity in Hilbert Spaces;

3. Attractive Points in Banach Spaces;

4. Nonlinear Ergodic Theorems in Banach Spaces.

Section 1

Nonlinear Ergodic Theorems
without Convexity in Hilbert Spaces

Let C be a nonempty subset of H.
For a mapping T of C into H, we denote
by $F(T)$ the set of all fixed points of T and
by $A(T)$ the set of all attractive points of T, i.e.,

1. $F(T) = \{x \in C : x = Tx\}$;

2. $A(T) = \{x \in H : \|Ty - x\| \leq \|y - x\|, \quad \forall y \in C\}$.

Lemma

C: a nonempty closed convex subset of H,
T: a mapping from C into itself.

If $A(T) \neq \phi$, then $F(T) \neq \phi$.

Lemma

C: a non-empty subset of H,
T: a mapping from C into H.

Then, $A(T)$ is a closed convex subset of H.

Theorem [Takahashi and Takeuchi, JNCA, 2011]

H: a real Hilbert space,
C: a nonempty subset of H.
T: a generalized hybrid mapping from C into itself.
Suppose that $A(T)$ is nonempty.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in A(T)$,
where $p = \lim_{n\to\infty} P_{A(T)} T^n x$.

Corollary

H: a real Hilbert space,
C: a nonempty subset of H.
T: a nonexpansive mapping from C into itself, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Suppose that $A(T)$ is nonempty.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in A(T)$,
where $p = \lim_{n\to\infty} P_{A(T)} T^n x$.

Corollary

H: a real Hilbert space,
C: a nonempty subset of H.
T: a nonspreading mapping from C into itself, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that $A(T)$ is nonempty.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in A(T)$,
where $p = \lim_{n \to \infty} P_{A(T)} T^n x$.

Corollary

H: a real Hilbert space,
C: a nonempty subset of H.
T: a hybrid mapping from C into itself, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that $A(T)$ is nonempty.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in A(T)$,
where $p = \lim_{n\to\infty} P_{A(T)} T^n x$.

Corollary [Kocourek, T. and Yao, Taiwanese J. Math., 2010]

C: a nonempty closed convex subset of H.
T: a generalized hybrid mapping from C into itself.
Suppose that $F(T)$ is nonempty.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in F(T)$,
where $p = \lim_{n\to\infty} P_{F(T)} T^n x$.

Section 2

Strong Convergence Theorems
without Convexity in Hilbert Spaces

C: a nonempty subset of a Hilbert space H.
Then, C is called *star-shaped*
if there exists $z \in C$ such that for any $x \in C$ and any $\lambda \in (0,1)$,

$$\lambda z + (1 - \lambda)x \in C.$$

Such $z \in C$ is called a *center* of star-shaped set C.

Theorem [Akashi and Takahashi, to appear]

H: a Hilbert space,
C: a star-shaped subset of H with center $z \in C$.
T: a nonexpansive mapping from C into itself with $A(T) \neq \emptyset$.
$\{x_n\}$: a sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n z + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongly to $u_0 \in A(T)$, where $u_0 = P_{A(T)} z$.

Using Theorem, we obtain
the following new fixed point theorem.

Theorem

H: a Hilbert space,
C: a closed and star-shaped subset of H.
T: a nonexpansive mapping from C into itself.
Suppose that there exists an $x \in C$ such that $\{T^n x\}$ is bounded.

Then $F(T) \neq \emptyset$.

Compare this theorem with
Itoh and Takahashi's fixed point theorem.

Theorem [Itoh and Takahashi, Pacific J. Math., 1978]
H: a Hilbert space,
C: a weakly compact and star-shaped subset of H.
T: a nonexpansive mapping from C into itself.

Then $F(T) \neq \emptyset$.

Theorem

H: a Hilbert space,
C: a closed and star-shaped subset of H with center $z \in C$.
T: a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$.
$\{x_n\}$: a sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n z + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongly to $u_0 \in F(T)$, where

$$\|u_0 - z\| = \min\{\|u - z\| : \ u \in F(T)\}.$$

Remark

(1) We do not know whether $F(T)$ is convex or not.
(2) There is no necessity for assuming that
C is star-shaped in Theorem.

Indeed, we can prove the following strong convergence theorem without convexity for nonexpansive mappings in a Hilbert space.

Theorem

H: a Hilbert space,
C: a nonempty subset of H.
T: a nonexpansive mapping from C into itself with $A(T) \neq \emptyset$.
$\{x_n\}$: a sequence defined by $z, x_1 \in C$ and

$$x_{n+1} = \alpha_n z + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

If $\{x_n\}$ is in C, then $\{x_n\}$ converges strongly to $u_0 \in A(T)$, where $u_0 = P_{A(T)} z$.

For example, let $\mathbb{R}^2$ be the 2-dimensional Euclidean space and $\mathbb{Q}$ be the set of all rational numbers.
Let $C = \mathbb{Q} \times \mathbb{Q}$
and let a mapping $T : C \to C$ be as follows:

$$Tx = (-b, a), \quad \forall x = (a, b) \in \mathbb{Q} \times \mathbb{Q}.$$

That is, T is a nonexpansive mapping of C into itself.
Furthermore, let $z, x_1 \in C$ and $\alpha_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$.
Then Theorem is available for such a setting.

Section 2

Problem:
Can we prove Akashi and Takahashi's theorem
for generalized hybrid mappings in a Hilbert space?

Section 3

Attractive Points in Banach Spaces

Section 3

E: a smooth Banach space.
J: the duality mapping of E, i.e.,

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

ϕ: $E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \forall x, y \in E.$$

We have

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{1}$$

for all $x, y, z \in E$.
From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$,
we can see that $\phi(x, y) \geq 0$.
Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2}$$

for $x, y, z, w \in E$.

Definition

E: a smooth Banach space.
C: a nonempty subset of E,
T: a mapping of C into E.

$A(T)$: the set of attractive points of T, i.e.,

$$A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x), \quad \forall x \in C\}.$$

Lemma

E: a smooth Banach space,
C: a nonempty subset of E.
T: a mapping from C into E.

Then $A(T)$ is a closed and convex subset of E.

Section 3

Let E be a smooth Banach space,
let C be a nonempty subset of E
let J be the duality mapping from E into E^*.
Then, a mapping $T : C \to E$ is called generalized nonspreading
if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ \leq \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}\end{aligned} \tag{3}$$

for all $x, y \in C$, where
$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$.
We call such a mapping
an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping.

Section 3

Let E be a smooth Banach space,
let C be a nonempty subset of E
let $T : C \to E$ be $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading, i.e.,
there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Tx, Ty) &+ (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\leq \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}\end{aligned}$$

for all $x, y \in C$.
If $\alpha = \gamma = \beta = 1$ and $\delta = 0$, then

$$\phi(Tx, Ty) + \phi(Ty, Tx) - \phi(Ty, x) \leq \phi(Tx, y).$$

That is, T is a nonspreading mapping in the sense of
Kohsaka and T. [SIAM J. Optimization and Arc. Math. 2008].

Furthermore, if E is a Hilbert space,
then we have $\phi(x,y) = \|x-y\|^2$ for $x,y \in E$.
So, we obtain the following:

$$\alpha\|Tx-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 + \gamma\{\|Tx-Ty\|^2 - \|x-Ty\|^2\} \\ \leq \beta\|Tx-y\|^2 + (1-\beta)\|x-y\|^2 + \delta\{\|Tx-y\|^2 - \|x-y\|^2\} \quad (4)$$

for all $x,y \in C$. This implies that

$$(\alpha+\gamma)\|Tx-Ty\|^2 + \{1-(\alpha+\gamma)\}\|x-Ty\|^2 \\ \leq (\beta+\delta)\|Tx-y\|^2 + \{1-(\beta+\delta)\}\|x-y\|^2 \quad (5)$$

for all $x,y \in C$. That is, T is a generalized hybrid mapping
in the sense of Kocourek, T. and Yao [Taiwanese J. Math., 2010]

We give the following defintition.
Let E be a smooth Banach space.
Let C be a nonempty subset of E
T be a mapping of C into E.
We denote by $B(T)$ the set of skew-attractive points of T, i.e.,

$$B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z), \quad \forall x \in C\}.$$

Section 4

Nonlinear Ergodic Theorems in Banach Spaces

Section 4

Theorem [Lin and Takahashi, J. Convex Anal., to appear]

E: a uniformly convex Banach space
with a Fréchet differentiable norm
C: a nonempty subset of E.
$T : C \to C$: a generalized nonspreading mapping such that
$A(T) = B(T) \neq \emptyset$.
R be the sunny generalized nonexpansive retraction of E onto $B(T)$.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $A(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Theorem [Kocourek, T. and Yao, Adv. Math. Econ., 2011]

E: a uniformly convex Banach space
with a Fréchet differentiable norm.
$T : E \to E$: $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading
such that $\alpha > \beta$, $\gamma \leq \delta$ and $F(T) \neq \emptyset$
R: the sunny generalized nonexpansive retraction of E onto $F(T)$.
Then, for any $x \in E$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $F(T)$,
where $q = \lim_{n \to \infty} RT^n x$.

Proof

We also know that $\alpha > \beta$ together with $\gamma \le \delta$ implies that

$$\phi(Tx, u) \le \phi(x, u)$$

for all $x \in E$ and $u \in F(T)$.

We also note that $A(T) = F(T)$ and $B(T) = F(T)$.

So, we have the desired result from Theorem.

Theorem [Takahashi and Takeuchi, JNCA, 2011]

H: a Hilbert space,
C: a nonempty subset of H.
$T : C \to C$: a generalized hybrid mapping with $A(T) \neq \emptyset$,
P: the mertic projection of H onto $A(T)$.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $A(T)$, where
$p = \lim_{n\to\infty} PT^n x$.

Proof
We first note that $A(T) = B(T)$.
Since $A(T) = B(T)$ is a nonempty closed convex subset of H,
there exists the metric projection of H onto $A(T)$.
In a Hilbert space,
the metric projection of H onto $A(T)$ is equivalent to
the sunny generalized nonexpansive retraction of E onto $A(T)$.
So, we have the desired result from Theorem.

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どうもありがとうございました!!

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