Fixed Point and Convergence Theorems for New Nonlinear Mappings

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Introduction

H: a Hilbert space,
C: a closed convex subset of H,
$f : C \times C \to \mathbb{R}$ satisfying (A1)–(A4):

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
(A3) $f(x,\cdot)$ is lower semicontinuous and convex for all $x \in C$;
(A4) $\lim_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$ for all $x, y, z \in C$.

Theorem [Blum and Oettli]

H: Hilbert space,
C: a closed convex subset of H,
f: $C \times C \to \mathbb{R}$ satisfying (A1)–(A4),
$r > 0$, $x \in H$.

Then, there exists a unique $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

H: a Hilbert space,
C: a closed convex subset of H,
f: $C \times C \to \mathbb{R}$ satisfying (A1)–(A4),
$r > 0$, $x \in H$.
$T_r : H \to C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}.$$

Then, T_r: firmly nonexpansive, i.e., for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle.$$

For $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$$
$$\iff 2\|T_r x - T_r y\|^2 \leq 2\langle T_r x - T_r y, x - y \rangle$$
$$\iff 2\|T_r x - T_r y\|^2 \leq \|x - y\|^2 + \|T_r x - T_r y\|^2 - \|x - y - (T_r x - T_r y)\|^2$$
$$\iff \|T_r x - T_r y\|^2 \leq \|x - y\|^2 - \|x - y - (T_r x - T_r y)\|^2$$
$$\Longrightarrow \|T_r x - T_r y\|^2 \leq \|x - y\|^2$$
$$\iff \|T_r x - T_r y\| \leq \|x - y\|.$$

$T : C \to H$: nonexpansive if
for all $x, y \in C$, $\|Tx - Ty\| \leq \|x - y\|$.

For $x, y, z, w \in H$,

$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$.

$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$
$\iff 2\|T_r x - T_r y\|^2 \leq \|x - T_r y\|^2 + \|y - T_r x\|^2 - \|x - T_r x\|^2 - \|y - T_r y\|^2$
$\implies 2\|T_r x - T_r y\|^2 \leq \|x - T_r y\|^2 + \|y - T_r x\|^2$.

$T : C \to H$: nonspreading if
for all $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.$$

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$$
$$\iff 4\|T_r x - T_r y\|^2 \leq 4\langle T_r x - T_r y, x - y \rangle$$
$$\iff 4\|T_r x - T_r y\|^2 \leq \|T_r x - T_r y\|^2 + \|x - y\|^2 - \|x - y - (T_r x - T_r y)\|^2$$
$$+ \|x - T_r y\|^2 + \|y - T_r x\|^2 - \|x - T_r x\|^2 - \|y - T_r y\|^2$$
$$\implies 3\|T_r x - T_r y\|^2 \leq \|x - y\|^2 + \|y - T_r x\|^2 + \|x - T_r y\|^2.$$

$T : C \to H$: hybrid if

for all $x, y \in C$,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2.$$

H: a Hilbert space,

C: a nonempty closed convex subset of H,

$T : C \to C$, nonexpansive

$\iff \|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$

Theorem [Baillon, 1975]

H: a Hilbert space,
C: a nonempty bounded closed convex subset of H,
$T : C \to C$, a nonexpansive mapping.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

Theorem [Takahashi and Yao, Taiwanese J. Math., 2010]

H: a Hilbert space,
C: a nonempty closed convex subset of H,
T: a mapping of C into itself such that $F(T)$ is nonempty.
Suppose that T satisfies one of the following conditions:
(i) T is nonspreading;
(ii) T is hybrid;
(iii) $2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C$.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

Definition [Aoyama, Iemoto, Kohsaka and Takahashi, JNCA, 2010]

H: a real Hilbert space,
C: a nonempty subset of H.
$\lambda \in \mathbb{R}$,
Then, $T : C \to H$, λ-hybrid if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C.$$

(1) $T : C \to H$: 0-hybrid if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2, \quad \forall x, y \in C,$$

i.e., T is nonexpansive.

(2) $T : C \to H$ is 1-hybrid if and only if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

i.e., T is nonspreading.

(3) $T : C \to H$ is $\frac{1}{2}$-hybrid if and only if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

i.e., T is hybrid.

Theorem [Aoyama, Iemoto, Kohsaka and Takahashi, JNCA, 2010]

H: a Hilbert space,
C: a nonempty closed convex subset of H,
$\lambda \in \mathbb{R}$,
$T : C \to C$, λ-hybrid, i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C.$$

Then, the following are equivalent:
(a) There exists $x \in C$ such that $\{T^n x\}$ is bounded;
(b) $F(T)$ is nonempty.

Theorem [Aoyama, Iemoto, Kohsaka and Takahashi, JNCA, 2010]

H: a Hilbert space,
C: a nonempty subset of H,
$\lambda \in [0, 1]$,
$T : C \to C$, firmly nonexpansive.
Then, $T : C \to C$ is λ-hybrid.

Theorem [Aoyama, Iemoto, Kohsaka and Takahashi, JNCA, 2010]

H: a Hilbert space,
C: a nonempty closed convex subset of H,
$\lambda \in [0, 1]$.
Then, the following are equivalent:

(i) Every λ-bybrid mapping of C into C has a fixed point;

(ii) C is bounded.

Theorem [Ray, Trans. Amer. Math. Soc., 1980]

H: a Hilbert space,
C: a nonempty closed convex subset of H.
Then, the following are equivalent:

(i) Every nonexpansive mapping of C into C has a fixed point;

(ii) C is bounded.

Theorem

H: a Hilbert space,
C: a nonempty closed convex subset of H.
Then, the following are equivalent:

(i) Every nonspreading mapping of C into C has a fixed point;

(ii) C is bounded.

Theorem [Takahashi, JNCA, 2010]

H: a Hilbert space,
C: a nonempty closed convex subset of H.
Then, the following are equivalent:

(i) Every hybrid mapping of C into C has a fixed point;

(ii) C is bounded.

Theorem [Aoyama, Iemoto, Kohsaka and Takahashi, JNCA, 2010]

H: a Hilbert space,
C: a nonempty closed convex subset of H,
$\lambda \in \mathbb{R}$,
$T : C \to C$, λ-hybrid, i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C.$$

If $F(T) \neq \emptyset$ and P is the metric projection H onto $F(T)$,
then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} PT^n x$.

Introduction

Our talk is organized as follows:

1. New Nonlinear Operators in Hilbert spaces;

2. Fixed Point Theorems;

3. Nonlinear Ergodic Theorems;

4. Weak and Strong Convergence Theorems.

Section 1

New Nonlinear Operators in Hilbert spaces

Definition [Kocourek, Takahashi and Yao, Taiwanese J. Math., 2010]

H: a real Hilbert space,
C: a nonempty subset of H.
Then, a mapping $T : C \to H$ is called *generalized hybrid* if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \quad (1)$$

for all $x, y \in C$.

T is called an (α, β)-*generalized hybrid* mapping.

(1) $T : C \to H$: $(1, 0)$-generalized hybrid if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2, \quad \forall x, y \in C,$$

i.e., T is nonexpansive.

(2) $T : C \to H$ is $(2, 1)$-generalized hybrid if and only if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

i.e., T is nonspreading.

(3) $T : C \to H$ is $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid if and only if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

i.e., T is hybrid.

Theorem [Hojo, Takahashi and Yao, to appear]

H: a Hilbert space,
C: a nonempty subset of H,
$\alpha, \beta \in \mathbb{R}$,
$T : C \to H$ is (α, β)-generalized hybrid
if and only if it satisfies that

$$\begin{aligned}\|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2 \\ + 2(\alpha - 1)\langle x - Tx, y - Ty\rangle - (\alpha - \beta - 1)\|y - Tx\|^2\end{aligned}$$

for all $x, y \in C$.

So, we have the following:

H: a Hilbert space,

C: a nonempty subset of H,

$\lambda \in \mathbb{R}$,

T: λ-hybrid, i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C.$$

Then, $T : C \to H$ is (α, β)-generalized hybrid with $\alpha - \beta = 1$.

Section 2

Fixed Point Theorems

Theorem [Takahashi and Yao, Taiwanese J. Math., 2010]

H: a Hilbert space,
C: a nonempty closed convex subset of H
T: a mapping of C into itself.
Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2, \quad \forall y \in C$$

for some Banach limit μ. Then, T has a fixed point in C.

Theorem [Kocourek, Takahashi and Yao, Taiwanese J. Math., 2010]

H: a Hilbert space,
C: a nonempty closed convex subset of H,
$T : C \to C$ be a generalized hybrid mapping.

Then T has a fixed point in C
if and only if $\{T^n z\}$ is bounded for some $z \in C$.

Proof

Since $T : C \to C$ is a generalized hybrid mapping,
there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \quad (2)$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then $\{T^n z\} = \{z\}$ for $z \in F(T)$.
So, $\{T^n z\}$ is bounded.

We show the reverse.
Take $z \in C$ such that $\{T^n z\}$ is bounded.
Let μ be a Banach limit.
Then, for any $y \in C$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} \alpha\|T^{n+1}z - Ty\|^2 &+ (1-\alpha)\|T^n z - Ty\|^2 \\ &\le \beta\|T^{n+1}z - y\|^2 + (1-\beta)\|T^n z - y\|^2 \end{aligned}$$

for any $y \in C$. Since $\{T^n z\}$ is bounded,
we can apply a Banach limit μ to both sides of the inequality.
Then, we have

$$\mu_n(\alpha\|T^{n+1}z - Ty\|^2 + (1-\alpha)\|T^n z - Ty\|^2) \\ \leq \mu_n(\beta\|T^{n+1}z - y\|^2 + (1-\beta)\|T^n z - y\|^2).$$

So, we obtain

$$\alpha\mu_n\|T^{n+1}z - Ty\|^2 + (1-\alpha)\mu_n\|T^n z - Ty\|^2 \\ \leq \beta\mu_n\|T^{n+1}z - y\|^2 + (1-\beta)\mu_n\|T^n z - y\|^2$$

and hence

$$\alpha\mu_n\|T^n z - Ty\|^2 + (1-\alpha)\mu_n\|T^n z - Ty\|^2 \\ \leq \beta\mu_n\|T^n z - y\|^2 + (1-\beta)\mu_n\|T^n z - y\|^2.$$

This implies

$$\mu_n \|T^n z - Ty\|^2 \leq \mu_n \|T^n z - y\|^2$$

for all $y \in C$.

By Takahashi and Yao's theorem, we have a fixed point in C.

Corollary [Kocourek, Takahashi and Yao, Taiwanese J. Math., 2010]

H: a Hilbert space,
C: a nonempty bounded closed convex subset of H,
T: a generalized hybrid mapping from C to itself.

Then T has a fixed point.

Further, we have the following:

Theorem [Aoyama, Iemoto, Kohsaka and Takahashi, JNCA, 2010]

H: a Hilbert space,
C: a nonempty closed convex subset of H,
$\lambda \in [0, 1]$.
Then, the following are equivalent:

(i) Every λ-bybrid mapping of C into C has a fixed point;

(ii) C is bounded.

Section 3

Nonlinear Ergodic Theorems

Theorem [Kocourek, Takahashi and Yao, Taiwanese J. Math., 2010]

H: a Hilbert space
C be a nonempty closed convex subset of H.
$T : C \to C$: a generalized hybrid mapping with $F(T) \neq \emptyset$
P: the mertic projection of H onto $F(T)$.
Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $F(T)$, where
$p = \lim_{n\to\infty} PT^n x$.

Proof

Since $T : C \to C$ is a generalized hybrid mapping,
there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \quad (3)$$

for all $x, y \in C$.
Since T is an (α, β)-generalized hybrid mapping,
T is quasi-nonexpansive.
So, we have that $F(T)$ is closed and convex.
Let $x \in C$ and let P be the metric projection of H onto $F(T)$.
Then, we have

$$\begin{aligned}\|PT^n x - T^n x\| &\le \|PT^{n-1} x - T^n x\| \\ &\le \|PT^{n-1} x - T^{n-1} x\|.\end{aligned}$$

This implies that $\{\|PT^n x - T^n x\|\}$ is nonincreasing.
We also know that for any $v \in C$ and $u \in F(T)$,

$$\langle v - Pv, Pv - u \rangle \geq 0$$

and hence

$$\|v - Pv\|^2 \leq \langle v - Pv, v - u \rangle.$$

So, we get

$$\begin{aligned} \|Pv - u\|^2 &= \|Pv - v + v - u\|^2 \\ &= \|Pv - v\|^2 - 2\langle Pv - v, u - v \rangle + \|v - u\|^2 \\ &\leq \|v - u\|^2 - \|Pv - v\|^2. \end{aligned}$$

Let $m, n \in \mathbb{N}$ with $m \geq n$.

Putting $v = T^m x$ and $u = PT^n x$, we have

$$\begin{aligned} \|PT^m x - PT^n x\|^2 &\leq \|T^m x - PT^n x\|^2 - \|PT^m x - T^m x\|^2 \\ &\leq \|T^n x - PT^n x\|^2 - \|PT^m x - T^m x\|^2. \end{aligned}$$

So, $\{PT^n x\}$ is a Cauchy sequence.
Since $F(T)$ is closed, $\{PT^n x\}$ converges strongly
to an element p of $F(T)$.
Take $u \in F(T)$. Then we obtain, for any $n \in \mathbb{N}$,

$$\|S_n x - u\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x - u\| \leq \|x - u\|.$$

So, $\{S_n x\}$ is bounded and hence
there exists a weakly convergent subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$.
If $S_{n_i} x \rightharpoonup v$, then we have $v \in F(T)$.

In fact, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have that

$$
\begin{aligned}
0 &\le \beta\|T^{k+1}x - y\|^2 + (1-\beta)\|T^k x - y\|^2 \\
&\quad - \alpha\|T^{k+1}x - Ty\|^2 - (1-\alpha)\|T^k x - Ty\|^2 \\
&= \beta\Big\{\|T^{k+1}x - Ty\|^2 + 2\left\langle T^{k+1}x - Ty, Ty - y\right\rangle + \|Ty - y\|^2\Big\} \\
&\quad + (1-\beta)\Big\{\|T^k x - Ty\|^2 + 2\left\langle T^k x - Ty, Ty - y\right\rangle + \|Ty - y\|^2\Big\} \\
&\quad - \alpha\|T^{k+1}x - Ty\|^2 - (1-\alpha)\|T^k x - Ty\|^2 \\
&= \|Ty - y\|^2 + 2\left\langle \beta T^{k+1}x + (1-\beta)T^k x - Ty, Ty - y\right\rangle \\
&\quad + (\beta - \alpha)\Big\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\Big\}.
\end{aligned}
$$

Summing up these inequalities with respect to $k = 0, 1, \ldots, n-1$,

$$0 \le n\|Ty - y\|^2 + 2\left\langle \sum_{k=0}^{n-1} T^k x + \beta(T^n x - x) - nTy, Ty - y \right\rangle$$
$$+ (\beta - \alpha)\Big\{\|T^n x - Ty\|^2 - \|x - Ty\|^2\Big\}.$$

Deviding this inequality by n, we have

$$0 \le \|Ty - y\|^2 + 2\left\langle S_n x + \frac{1}{n}\beta(T^n x - x) - Ty, Ty - y \right\rangle$$
$$+ \frac{1}{n}(\beta - \alpha)\Big\{\|T^n x - Ty\|^2 - \|x - Ty\|^2\Big\},$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Replacing n by n_i and letting $n_i \to \infty$, we obtain from $S_{n_i} x \rightharpoonup v$ that

$$0 \le \|Ty - y\|^2 + 2\langle v - Ty, Ty - y \rangle.$$

Putting $y = v$, we have $0 \leq -\|Tv - v\|^2$ and hence $Tv = v$.
To complete the proof, it is sufficient to show that
if $S_{n_i}x \rightharpoonup v$, then $v = p$. We have that

$$\langle T^k x - PT^k x, PT^k x - u\rangle \geq 0$$

for all $u \in F(T)$. Since $\{\|T^k x - PT^k x\|\}$ is nonincreasing,
we have

$$\begin{aligned}\langle u - p, T^k x - PT^k x\rangle &\leq \langle PT^k x - p, T^k x - PT^k x\rangle \\ &\leq \|PT^k x - p\| \cdot \|T^k x - PT^k x\| \\ &\leq \|PT^k x - p\| \cdot \|x - Px\|.\end{aligned}$$

Adding these inequalities from $k = 0$ to $k = n - 1$ and dividing n,
we have

$$\langle u - p, S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x\rangle \leq \frac{\|x - Px\|}{n}\sum_{k=0}^{n-1}\|PT^k x - p\|.$$

Since $S_{n_i}x \rightharpoonup v$ and $PT^k x \to p$, we have

$$\langle u - p, v - p \rangle \leq 0.$$

We know $v \in F(T)$. So, putting $u = v$,
we have $\langle v - p, v - p \rangle \leq 0$ and hence $\|v - p\|^2 \leq 0$.
So, we obtain $v = p$.
This completes the proof.

Section 3

From Kocourek, Takahashi and Yao's nonlinear ergodic theorem, we have:

Theorem [Aoyama, Iemoto, Kohsaka and Takahashi, JNCA, 2010]
H: a Hilbert space,
C: a nonempty closed convex subset of H,
$\lambda \in \mathbb{R}$,
$T : C \to C$, λ-hybrid, i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C.$$

If $F(T) \neq \emptyset$ and P is the metric projection H onto $F(T)$, then, for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} PT^n x$.

Section 4

Weak and Strong Convergence Theorems

Lemma

H: a Hilbert space,

C: a nonempty closed convex subset of H.

$T : C \to C$: a generalized hybrid mapping.

Then, $I - T$ is demiclosed, i.e.,

$x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$ imply $z \in F(T)$.

Theorem [Kocourek, Takahashi and Ya, Taiwanese J. Math., 2010]

H: a Hilbert space,
C: a closed convex subset of H.
$T : C \to C$: a generalized hybrid mapping with $F(T) \neq \emptyset$,
P: the mertic projection of H onto $F(T)$.
$\{\alpha_n\}$: a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.
Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n = 1, 2, \ldots.$$

Then, the sequence $\{x_n\}$ converges weakly to an element v of $F(T)$, where $v = \lim_{n\to\infty} Px_n$.

Theorem [Kocourek, Takahashi and Ya, Taiwanese J. Math., 2010]
H: a Hilbert space,
C: a nonempty closed convex subset of H.
γ: a real number with $\gamma \neq -1$,
$S : C \to H$: a mapping such that

$$\|Sx - Sy\|^2 + 2\gamma\langle x - y, Sx - Sy\rangle \leq (1 + 2\gamma)\|x - y\|^2,\ \forall x, y \in C.$$

$\{\alpha_n\} \subset [0, 1]$: a sequence of real numbers such that $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.
Suppose $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_C\Big\{\frac{1}{1+\gamma}Sx_n + \frac{\gamma}{1+\gamma}x_n\Big\}, \quad n = 1, 2, \ldots.$$

If $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element v of $F(S)$, where $v = P_{F(S)}u$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Proof We have that for any $x, y \in C$,

$$
\begin{aligned}
&\|Sx - Sy\|^2 + 2\gamma\langle x - y, Sx - Sy\rangle \leq (1 + 2\gamma)\|x - y\|^2 \\
\iff &\|Sx - Sy\|^2 + \gamma(\|x - Sy\|^2 + \|Sx - y\|^2 - \|Sx - x\|^2 - \|y - Sy\|^2) \\
&\qquad \leq (1 + 2\gamma)\|x - y\|^2 \\
\iff &\|Sx - Sy\|^2 + \gamma\|x - Sy\|^2 \\
&\qquad \leq -\gamma\|Sx - y\|^2 + (1 + 2\gamma)\|x - y\|^2 + \gamma\|Sx - x\|^2 + \gamma\|y - Sy\|^2.
\end{aligned}
$$

So, S is a (1, 0, γ)-super hybrid mapping of C into H.
Put $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, we have
that T is a (1, 0)-generalized hybrid mapping of C into H, i.e.,
T is a nonexpansive mapping of C into H.
Furthermore, we have $F(S) = F(T)$.
From Wittmann's theorem, we obtain $x_n \to P_{F(P_C T)}u$.
Let us show $F(P_C T) = F(T) = F(S)$.

We know $F(T) = F(S)$. It is obvious that $F(T) \subset F(P_C T)$.
We show $F(P_C T) \subset F(T)$. If $P_C T v = v$,
we have from the property of P_C that for $u \in F(T)$,

$$\begin{aligned} 2\|v-u\|^2 &= 2\|P_C Tv - u\|^2 \\ &\leq 2\langle Tv - u, P_C Tv - u\rangle \\ &= \|Tv-u\|^2 + \|P_C Tv - u\|^2 - \|Tv - P_C Tv\|^2 \end{aligned}$$

and hence

$$2\|v-u\|^2 \leq \|v-u\|^2 + \|v-u\|^2 - \|Tv-v\|^2.$$

So, we have $0 \leq -\|Tv-v\|^2$ and hence $Tv = v$.
This completes the proof.

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Thank You Very Much!!

非常感謝!!

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