Using Aleksandrov Reflection to Estimate the Location of the Center of Expansion

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Introduction

Let γ_0 be smooth convex embedded closed curve in $\mathbb{R}^2$ parametrized by $X_0(\alpha)$, $\alpha \in S^1$.
Consider the expanding flow

$$\begin{cases} \frac{\partial \gamma}{\partial t}(\alpha, t) = G\left(\frac{1}{k(\alpha,t)}\right) N(\alpha, t), \quad \alpha \in S^1, \quad t > 0, \\ \gamma(\alpha, 0) = X_0(\alpha), \qquad \alpha \in S^1, \end{cases} \qquad (*)$$

where
$N(\alpha, t)$: the unit outward normal to $\gamma(\alpha, t)$

$k(\alpha, t)$: the curvature

$G : (0, \infty) \to (0, \infty)$ is a smooth function with $G' > 0$ everywhere (**parabolicity condition**).

By two formulae concerning length $L(t)$ and enclosed area $A(t)$:

$$\frac{dL}{dt}(t) = \int_{\gamma_t} \left\langle \frac{\partial \gamma}{\partial t}, kN \right\rangle ds \quad \text{and} \quad \frac{dA}{dt}(t) = \int_{\gamma_t} \left\langle \frac{\partial \gamma}{\partial t}, N \right\rangle ds,$$

we have

$$\begin{aligned} & \frac{d}{dt}\left[L^2(t) - 4\pi A(t)\right] \\ = \;& 2\left[\int_{\gamma_t} ds \int_{\gamma_t} kG\left(\frac{1}{k}\right) ds - \int_{\gamma_t} kds \int_{\gamma_t} G\left(\frac{1}{k}\right) ds\right] \le 0 \end{aligned}$$

since $G' > 0$.

If $A(t) \to \infty$ as $t \to T_{\max}$, then by the Bonnesen inequality

$$\frac{L^2(t)}{4\pi A(t)} - 1 \ge \pi \left(1 - \frac{r_{in}(t)}{r_{out}(t)}\right)^2$$

where $r_{in}(t)$ and $r_{out}(t)$ are respectively the radii of the largest inscribed circle and the smallest circumscribed circle of the curve $\gamma(\cdot, t)$, and so $r_{in}(t) / r_{out}(t) \to 1$ as $t \to \infty$. That is, γ_t evolves to become more and more circular.

Goal : Under the mild assumption on the speed function G, we want to describe the asymptotic shape and location of γ_t. In fact,

- γ_t will asymptotically look like the expanding circes $C(\cdot, t)$ centered at some point (a, b).
- Does the center (a, b) lie in the interior of γ_0?
  Yes. Under the assumption, we will show that the center lies in the interior of γ_0.
- Finally, we will give an example to demonstrate that if the assumption of the speed function is not satisfied, the the center of expansion may not exist in general.

Definition of the support function U

Let $\gamma(\varphi)$, $\varphi \in I$, be a smooth curve plane curve. The support function is defined by

$$u(\varphi) := \langle \gamma(\varphi), N_{out}(\varphi) \rangle, \ \varphi \in I.$$

- For a convex closed curve γ_0, in terms of its outward normal angle $\theta \in S^1$, u is defined as

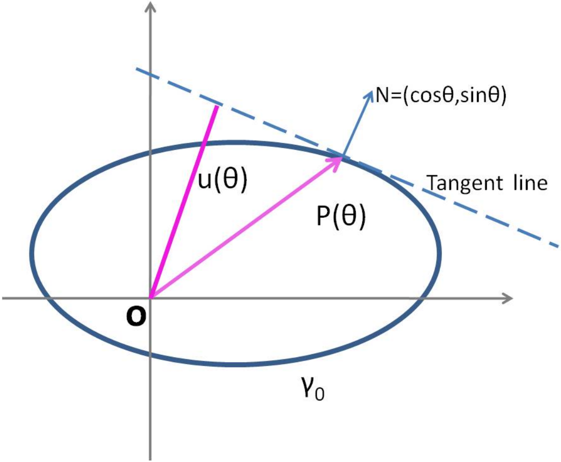
$$u(\theta) = \langle P(\theta), (\cos\theta, \sin\theta) \rangle, \quad \theta \in S^1 \tag{1}$$

where $P(\theta)$ is the position vector of the unique point $p \in \gamma_0$ whose outward normal angle is θ. Moveover the curveture $k_0(\theta)$ can be expressed as

$$k_0(\theta) = \frac{1}{u_{\theta\theta}(\theta) + u(\theta)} > 0, \quad \text{for all} \ \ \theta \in S^1. \tag{2}$$

- Note that γ_0 is a circle with radius R centered at $(a, b) \in \mathbb{R}^2$ if and only if

$$u_0(\theta) = R + a\cos\theta + b\sin\theta, \quad \theta \in S^1.$$



In terms of the *support function* $u(\theta, t)$ of $\gamma(\cdot, t)$, where the parameter $\theta \in S^1$ represents the outward normal angle, flow $(*)$ is equivalent to the following scalar equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = G\left(u_{\theta\theta} + u\right), & \theta \in S^1, \quad t > 0 \\ u(\theta, 0) = u_0(\theta) > 0, & \theta \in S^1. \end{cases} \tag{3}$$

together with the condition

$$u_{\theta\theta}(\theta, t) + u(\theta, t) > 0, \tag{4}$$

whenever the solution exists.

Assume $(u_0)_{\theta\theta}(\theta) + u(\theta) \geq \delta > 0$ for all $\theta \in S^1$.

Theorem (Chow-Tsai, [CT])

There exists a unique solution $u \in C^\infty\left(S^1 \times [0, T_{\max})\right)$ *to the equation* (3) *satisfying* (4), *where* $0 < T_{\max} \leq \infty$, *such that* $\lim_{t \to T_{\max}} u_{\min}(t) = \infty$.

Futhermore,

(i) $u_{\theta\theta} + u \geq \delta > 0$ *on* $S^1 \times [0, T_{\max})$.

(ii) u_θ *and* $u_{\theta\theta}$ *are uniformly bounded.*

(iii) *there exists a constant* $C > 0$ *such that*

$$0 \leq u_{\max}(t) - u_{\min}(t) < C, \ \ t \in [0, T_{\max}).$$

Theorem (Chow-Tsai, [CT])

There exists a solution $R(t)$ to the ODE $dR/dt = G(R)$ on $[0, T_{\max})$ such that

$$u_{\min}(t) \le R(t) \le u_{\max}(t),$$

and the support function $\tilde{u} = u/R$ to the rescaled curves $\tilde{\gamma} = \gamma/R$ satisfies

$$\|\tilde{u} - 1\|_{C^2(S^1)} \le \frac{C}{R(t)}, \quad t \in [0, T_{\max}).$$

Furthermore, if $\lim_{z\to\infty} G(z) = \infty$, then such an $R(t)$ is unique.

The shapes of the curves become round asymptotically in the sense that if one rescales the equation appropriately, the support functions of the rescaled curves converge uniformly to the constant function 1 in C^2-norm.

In this talk, we want to show that under a mild assumption on the speed function G, there exists a center of expansion where we expand a convex embedded closed curve in $\mathbb{R}^2$; that is, (**without rescaling**) the expanding curve $\gamma(\cdot, t)$ asymptotically looks like the expanding circles $C(\cdot, t)$ centered at some point (a, b) in C^1-norm.
From now on, we assume the speed function G satisfies the following assumption:

Main assumptions $(*)$ on G. We assume

$(*1)$: $\lim_{z\to\infty} G(z) = \infty$, and
$(*2)$: For any constant $C > 0$, there exists a constant $\lambda > 0$ such that

$$0 < \frac{1}{\lambda} \leq \frac{G'(\xi)}{G'(z)} \leq \lambda \quad \text{for all} \quad \xi \in [z - C,\ z + C] \tag{5}$$

as long as z is large enough.

Existence of the Center of Expansion

Remark

- *One can check that if there exists a number* $z_0 > 0$ *so that* $\log G'(z)$ *is uniformly continuous on* $[z_0, \infty)$, *then the condition* $(*2)$ *is satisfied.*
- *Examples for* $G(z)$ *satisfying* $(*2)$ *include* $G(z) = z^{\alpha}$, $G(z) = [\log(z+1)]^{\alpha}$, $G(z) = e^{\alpha z}$, *where* $\alpha > 0$ *is any constant, and many more.*

Lemma

Assume $(*1)$, $(*2)$ *and that the convex closed initial smooth curve* γ_0 *encloses the origin. Then under flow* $(*)$, *the support function* $u(\theta, t) \in C^{\infty}(S^1 \times [0, T_{\max}))$ *of* $\gamma(\cdot, t)$ *satisfies the estimate*

$$\lim_{t \to T_{\max}} \left\| u(\theta, t) - (R(t) + a\cos\theta + b\sin\theta) \right\|_{C^1(S^1)} = 0 \qquad (6)$$

Remark

- *Notice the relation between the position vector* $P(\theta)$ *and the support function* $U(\theta)$

$$P(\theta) = U(\theta)(\cos\theta, \sin\theta) + U_\theta(\theta)(-\sin\theta, \cos\theta) \tag{7}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} P(\theta)\, d\theta = \frac{1}{\pi}\int_0^{2\pi} U(\theta)(\cos\theta, \sin\theta)\, d\theta. \tag{8}$$

- *Due to* (7), **the geometric meaning** *of* (6) *is that as* $\gamma(\cdot, t)$ *expands to infinity, its support function is close to that of the expanding circle* $C(\cdot, t)$, *where* $C(\cdot, t)$ *is centered at* $(a, b) \in \mathbb{R}^2$ *with radius* $R(t)$, $t \in [0, T_{\max})$.

Remark

- *Due to* (8) *and* (6) ,

$$
\begin{aligned}
(a, b) &= \lim_{t \to T_{\max}} \frac{1}{\pi} \int_0^{2\pi} u(\theta, t) (\cos\theta, \sin\theta)\, d\theta \\
&= \lim_{t \to T_{\max}} \frac{1}{2\pi} \int_0^{2\pi} P(\theta, t)\, d\theta.
\end{aligned} \tag{9}
$$

- *(***Important***) Under the flow* $(*)$*, the isoperimetric difference* $L^2(t) - 4\pi A(t)$ *of* $\gamma(\cdot, t)$ *is always decreasing for* $t \in [0, T_{\max})$*. Moreover, if the flow* $(*)$ *has a center of expansion, then*

$$
L^2(t) - 4\pi A(t) \to 0 \quad as \quad t \to T_{\max}.
$$

Sketch the proof of Lemma:

- The key poins are the following:
  - $(*1)$ guaratees the ODE solution $R(t)$ is unique and the time interval for the rescaled new time τ is infinite so that we have enough time to establish stabilization to $a\cos\theta + b\cos\theta$ $(\text{as } \tau \to \infty)$.
  - $(*2)$ implies that the rescaled equation is uniformly parabolic.
- Set $w(\theta, t) = u(\theta, t) - R(t)$, $(\theta, t) \in S^1 \times [0, T_{\max})$ where $R(t)$ is from Theorem ([CT]).
- The evolution equation of $w(\theta, t)$ is

$$\begin{cases} \frac{\partial w}{\partial t}(\theta, t) = a(\theta, t)\left[w_{\theta\theta}(\theta, t) + w(\theta, t)\right], \\ a(\theta, t) = \int_0^1 G'\left(s\left(w_{\theta\theta} + w\right)(\theta, t) + R(t)\right) ds > 0. \end{cases} \tag{10}$$

- To make (10) uniformly parabolic, recale time by setting

$$\tau(t) = \log\left[G(R(t))/G(R(0))\right] \in [0,\infty),\ t \in [0, T_{\max}).$$

$\Rightarrow \tau(T_{\max}) = \infty$ (since G satisfies $(*)$ 1.) and (10) becomes

$$\begin{cases} \frac{\partial w}{\partial \tau}(\theta,\tau) = A(\theta,\tau)\left[w_{\theta\theta}(\theta,\tau) + w(\theta,\tau)\right], \ (\theta,\tau) \in S^1 \times [0,\infty) \\ A(\theta,\tau) = \frac{1}{G'(R(t))}\int_0^1 G'\left(s\left(w_{\theta\theta}+w\right)(\theta,t) + R(t)\right) ds \end{cases} \tag{11}$$

with

$$0 < \frac{1}{\lambda} \le A(\theta,\tau) \le \lambda\ , \ \text{(uniformly parabolic)}$$

$$|w(\theta,\tau)|, \quad |w_\theta(\theta,\tau)|, \quad |w_{\theta\theta}(\theta,\tau)|, \quad \left|\frac{\partial w}{\partial \tau}(\theta,\tau)\right| \le C$$

where C is a positive constant independent of (θ,τ).

- To (11), we want to show that there exist a, $b \in \mathbb{R}$ such that

$$\|w(\theta,\tau) - a\cos\theta - b\sin\theta\|_{C^1(S^1)} \to 0 \quad \text{as} \quad \tau \to \infty.$$

- let Y be the function space $\left\{c_1 \cos\theta + c_2 \sin\theta : c_1,\ c_2 \in \mathbb{R},\ \theta \in S^1\right\}$ and set

$$\rho(\tau) = \inf_{v \in Y} \|w(\cdot,\tau) - v\|_{L^2(S^1)}, \quad E(\tau) = \frac{1}{2}\int_0^{2\pi} \left(\left(\frac{\partial w}{\partial \theta}\right)^2 - w^2\right) d\theta, \tag{12}$$

$\Rightarrow E(\tau)$ is non-increasing in $\tau \in [0,\infty)$ and $\lim_{\tau\to\infty} \rho(\tau) = 0$.
$\Rightarrow$ there are bounded functions $a(\tau)$, $b(\tau)$ such that $\|w(\theta,\tau) - a(\tau)\cos\theta - b(\tau)\sin\theta\|_{L^2(S^1)} \to 0$ as $\tau \to \infty$.

- Moreover, since $w_{\theta\theta}(\theta,\tau)$ is uniformly bounded, this convergence is actually valid in the space $C^1(S^1)$.
- Finally denote by $\mathcal{Z}[u]$ the *number of sign changes* of a function $u(\theta)$ on S^1. For any $v \in Y$, the number $\mathcal{Z}[w(\cdot,\tau) - v]$ is *non-increasing* in $\tau \in (0,\infty)$ since $w(\theta,\tau) - v(\theta)$ is also a solution to equation (11). As a result of it, we can infer the convergence of $a(\tau)$ and $b(\tau)$ to some constants a and b respectively as $\tau \to \infty$. The proof is done. □

The location of the center of expansion

Next, we want to estimate the location of the center (a, b).
Notation:

- Let L be a line perpendicular to a unit vector $V \in \mathbb{R}^2$. Hence there is a constant C such that $\langle L, V\rangle = C$, where $\langle , \rangle$ is the inner product of $\mathbb{R}^2$. Denote by

$$H_+(L) = \{p \in \mathbb{R}^2 : \langle p, V\rangle > C\}$$

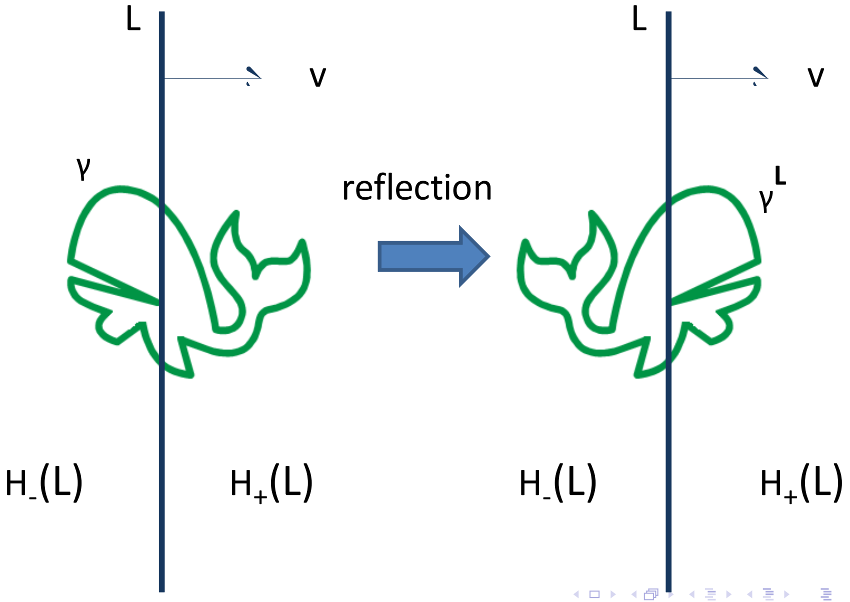
  and

$$H_-(L) = \{p \in \mathbb{R}^2 : \langle p, V\rangle < C\}.$$

  They are half-planes to both sides of L.

- Let $\gamma \subset \mathbb{R}^2$ be an embedded smooth closed curve and let γ^L be the *reflection* of γ about the line L, i.e.,

$$\gamma^L = \{p - 2(\langle p, V\rangle - C)V : p \in \gamma\}. \tag{13}$$


L
v
γ
H-(L)
H+(L)
reflection
L
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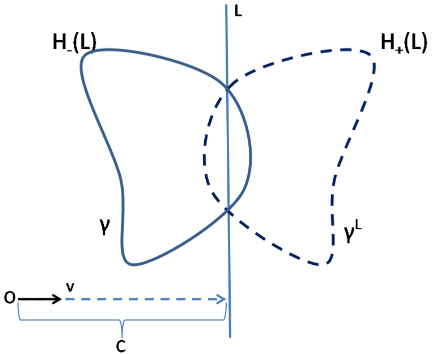
γ^L
H-(L)
H+(L)

Definition

We say that we can *reflect* γ *strictly at* (L, V) if

$$\gamma^L \cap H_-(L) \subset int(\gamma) \cap H_-(L) \tag{14}$$

and $V \notin T\gamma_p$ (the tangent space to γ at p) for any $p \in \gamma \cap L$. Here $int(\gamma)$ denotes the plane region interior to γ.



Definition

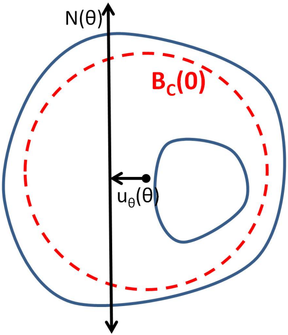
We say that we can *reflect* γ *strictly up to* (L, V) if we can reflect γ strictly at (L', V), where L' is any line parallel to L such that $\langle L', V \rangle \geq C$. In particular, this implies $V \notin T\gamma_p$ for any $p \in \gamma \cap \overline{H_+(L)}$, where $\overline{H_+(L)}$ denotes the closure of the region $H_+(L)$.

Theorem (*Chow-Gulliver [CG]*)

Let $G : (0, \infty) \to (0, \infty)$, $G' > 0$, *be an arbitrary smooth function (here it may not satisfy* $(*1)$ *and* $(*2)$*). For the flow* $(*)$*, if we can reflect the convex* γ_0 *strictly at (respectively, up to)* (L, V), *then we can reflect* $\gamma(\cdot, t)$ *strictly at (respectively, up to)* (L, V) *for all time* $t \in [0, T_{\max})$.

Remark

The Aleksandrove reflection result of Chow-Gulliver provides an elegant geometric proof of the gradient estimate (see [C]). Let $B_C(0)$ *be a circle centered at* $(0,0)$ *with radius* R *satisfying* γ_0 *is contained in the interior of* $B_C(0)$. *Then for* $x \in \gamma_t$, $t \in [0, T_{\max})$, $x_\perp := x - \langle x, N(x) \rangle$ *and* $|x_\perp| \le C$.
In terms of normal angle θ, $|u_\theta(\theta, t)| = |x_\perp| \le C$.



- Since γ_0 is strictly convex and smooth, along each normal direction $\mathbf{N}(\theta) = (\cos\theta, \sin\theta)$, $\theta \in S^1$, there exists a *unique* line $L(\theta)$ perpendicular to $\mathbf{N}(\theta)$, with $L(\theta) \cap int(\gamma_0) \neq \varnothing$, such that we can reflect γ_0 *strictly up to* $(L'(\theta), \mathbf{N}(\theta))$ where $L'(\theta)$ is any line parallel to $L(\theta)$ such that $\langle L'(\theta), \mathbf{N}(\theta)\rangle > \langle L(\theta), \mathbf{N}(\theta)\rangle$.

- Using these lines one can determine a region Ω strictly interior to γ_0, given by

$$\Omega = \left\{ p \in \mathbb{R}^2 : \langle p, \mathbf{N}(\theta)\rangle \leq \langle L(\theta), \mathbf{N}(\theta)\rangle \text{ for all } \theta \in S^1 \right\}. \quad (15)$$

This region is *convex.*

We can now prove the following:

Theorem

Let $G : (0, \infty) \to (0, \infty)$, $G' > 0$, *be an arbitrary smooth function satisfying* $(*)$ *(so that the center exists). Then under the flow* $(*)$*, the center of expansion lies on* Ω.

Proof.

- For a fixed $\theta_0 \in S^1$, we may assume $\mathbf{N}(\theta_0) = (0, 1)$ and the unique line $L(\theta_0)$ is the line $y = 0$ (i.e., x-axis). Also choose the origin of $\mathbb{R}^2$ at some point $O \in L(\theta_0) \cap int(\gamma_0)$.

- By Theorem ([CG]), for any $\varepsilon > 0$ we have

$$\gamma^{\tilde{L}}(\cdot, t) \cap H_-(\tilde{L}) \subset int(\gamma(\cdot, t)) \cap H_-(\tilde{L}) \tag{16}$$

  where $\tilde{L}$ is the line $\tilde{L} = \{y = \varepsilon > 0\}$ and moreover $(0, 1) \notin T\gamma(\cdot, t)_p$ for all $p \in \gamma(\cdot, t) \cap \overline{H_+(\tilde{L})}$ for all $t \in [0, T_{\max})$.

By the inclusion relation (16) we have

$$\left\langle \frac{1}{2\pi}\int_0^{2\pi}\left[X\left(\theta,t\right)-\left(0,\varepsilon\right)\right]d\theta,\ \left(0,1\right)\right\rangle \tag{17}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left\langle X\left(\theta,t\right)-\left(0,\varepsilon\right),\ \left(0,1\right)\right\rangle d\theta \leq 0 \tag{18}$$

for all $t \in [0, T_{\max})$, where the position vector $X(\theta, t)$ is with respect to the origin O. Letting $t \to T_{\max}$ and $\varepsilon \to 0$ in (17), by (9) we obtain $\langle (a, b), (0, 1)\rangle \leq 0$, which means that the center (a, b) is on or below the x-axis. As this property is valid for any direction $\theta_0 \in S^1$, we must have $(a, b) \in \Omega$. □

Nonexistence of Center of Expansion

Let K and L be two compact subsets in $\mathbb{R}^n$. Their *Hausdorff distance* is defined as

$$\delta(K,L) = \max\left\{ \sup_{\theta\in K} \inf_{y\in L} |\theta - y|, \quad \sup_{\theta\in L} \inf_{y\in K} |\theta - y| \right\}. \tag{19}$$

We have the following result (see Schneider [S], p. 53):

Lemma

Assume K and L are two convex bodies in $\mathbb{R}^n$, then

$$\delta(K,L) = \sup_{\theta\in S^{n-1}} |U_K(\theta) - U_L(\theta)| \tag{20}$$

where $U_K(\theta)$ and $U_L(\theta)$ are the support function of K and L respectively.

For convex bodies, the enclosed area A and length L are continuous functionals in the Hausdorff distance.

Now we will construct an example for the smooth speed function $G:(0,\infty)\rightarrow(0,\infty)$, $G'>0$ everywhere which does not satisfy the main assumption and show that the center of expansion does not exist.

- Recall we have known that
  if the center of expansion exists $\Rightarrow$ $L^2(t)-4\pi A(t)\rightarrow 0$ as $t\rightarrow T_{\max}$.
  Thus, if $L^2(t)-4\pi A(t)\not\rightarrow 0$ as $t\rightarrow T_{\max}$, then the center of expansion does not exists.
- Note that the existence of the limit

$$\lim_{t\rightarrow T_{\max}}\frac{1}{\pi}\int_0^{2\pi}u(\theta,t)(\cos\theta,\sin\theta)\,d\theta=(a,b)\in\mathbb{R}^2 \qquad (21)$$

  does *not* necessarily imply that $(a,b)\in\mathbb{R}^2$ is the center of expansion since it may not imply that $u(\theta,t)$ is asymptotically given by $R(t)+a\cos\theta+b\sin\theta$. From the viewpoint of Fourier series expansion, it could contain terms of the form $a_n\cos n\theta+b_n\sin n\theta$, $n\geq 2$, $n\in\mathbb{N}$.

- Take $G(z) = 1 - e^{-z}$, $z \in (0, \infty)$, $G' > 0$ everywhere.
  $\Rightarrow G$ satisfies condition $(*2)$ but does not satisfy condition $(*1)$.

- Under the flow $(*)$, the evolution equation of the support function $u(\theta, t)$ is

$$\frac{\partial u}{\partial t}(\theta, t) = 1 - e^{-(u_{\theta\theta}+u)}. \tag{22}$$

- Let $r(t)$ be the solution to the ODE:

$$\frac{dr}{dt}(t) = 1 - e^{-r(t)}, \ r(0) = 1.$$

$\Rightarrow \ r(t)$ is defined on $[0, \infty)$, $r(\infty) = \infty$.

- The rescaled new time

$$\tau(t) = \log\left(\frac{1-e^{-r(t)}}{1-e^{-r(0)}}\right) \in [0,\lambda), \quad \lambda = \log\left(\frac{e}{e-1}\right) < \infty.$$

- Let $w(\theta,t) = u(\theta,t) - r(t)$. Then in terms of the new time τ it satisfies the equation

$$\frac{\partial w}{\partial \tau}(\theta,\tau) = \frac{\left[1-e^{-(w_{\theta\theta}(\theta,\tau)+w(\theta,\tau)+r(t))}\right]-\left[1-e^{-r(t)}\right]}{e^{-r(t)}} \quad (23)$$

$$= 1-e^{-[w_{\theta\theta}(\theta,\tau)+w(\theta,\tau)]} \quad (24)$$

which happens to be the same as the original equation (22).

- For a given sufficiently small $\varepsilon > 0$, choose $T > 0$ such that $\tau(T) = \lambda - \varepsilon/50$ and let

$$\begin{aligned} w^*(\theta,\tau) &= \cos 2\theta + 100\left[\tau - \left(\lambda - \frac{\varepsilon}{50}\right)\right], \\ w_*(\theta,\tau) &= \cos 2\theta - 100\left[\tau - \left(\lambda - \frac{\varepsilon}{50}\right)\right] \end{aligned}$$

where $(\theta,\tau) \in S^1 \times [\lambda - \varepsilon/50, \lambda) := I$. We have

$$\left|1 - e^{-\left[w^*_{\theta\theta}(\theta,\tau) + w^*(\theta,\tau)\right]}\right| \leq 100 \quad \text{and} \quad \left|1 - e^{-\left[w_{*\theta\theta}(\theta,\tau) + w_*(\theta,\tau)\right]}\right| \leq 100,$$

Hence w^* and w_* are both super-solution and sub-solution to the equation (23) on $(\theta,\tau) \in I$.

- Let $u(\theta,t)$ be the solution to equation

$$\begin{cases} \frac{\partial u}{\partial t}(\theta,t) = 1 - e^{-(u_{\theta\theta}+u)}, & \theta \in S^1, \quad t \in (T,\infty) \\ u(\theta,T) = r(T) + \cos 2\theta, & \theta \in S^1 \end{cases}$$

- The geometric meaning: at time T one can use the support function $u(\theta, T) = r(T) + \cos 2\theta$ to construct a smooth convex closed curve $\gamma(\cdot, T)$ with curvature

$$k(\theta, T) = \frac{1}{u_{\theta\theta}(\theta, T) + u(\theta, T)} = \frac{1}{r(T) - 3\cos 2\theta} > 0, \quad \theta \in S^1$$

and evolve it under the expanding flow.

- Then the function $w(\theta, t) = u(\theta, t) - r(t)$, $t \in [T, \infty)$, in terms of the variable (θ, τ), is a solution to (23)
on $(\theta, \tau) \in I$ with $w(\theta, \lambda - \varepsilon/50) = \cos 2\theta$ and by the maximum principle we obtain

$$\cos 2\theta - 2\varepsilon \le w_*(\theta, \tau) \le w(\theta, \tau) \le w^*(\theta, \tau) \le \cos 2\theta + 2\varepsilon$$

on I, which implies that

$$(r(t) + \cos 2\theta) - 2\varepsilon \le u(\theta, t) \le (r(t) + \cos 2\theta) + 2\varepsilon \qquad (25)$$

on $(\theta, t) \in S^1 \times [T, \infty)$.

- We choose ε to be sufficiently smaller, then T will become larger and $u(\theta, t)$ will be closer to $r(t) + \cos 2\theta$ on $S^1 \times [T, \infty)$.
  $\Rightarrow$ the isoperimetric difference $L^2 - 4\pi A$ of $\gamma(\cdot, t)$ will be close to the isoperimetric difference of the convex curve with support function $r(t) + \cos 2\theta$ for $t \in [T, \infty)$.
- However, the isoperimetric difference of the latter is a *fixed* positive constant independent of time, given by
  $L^2(t) - 4\pi A(t)$
  $$= \left(\int_0^{2\pi} \cos 2\theta d\theta \right)^2 - 2\pi \int_0^{2\pi} \left[(\cos 2\theta)^2 - (\cos 2\theta)_\theta^2 \right] d\theta = 6\pi^2,$$
  for all $t \in [T, \infty)$.
- Hence **the isoperimetric difference of** $\gamma(\cdot, t)$ **is closer to the constant** $6\pi^2$ **as** $t \to \infty$ **and there is no center of expansion for** $\gamma(\cdot, t)$**.** This also says that if $u(\theta, t)$ satisfies (25) on $S^1 \times [T, \infty)$, then it is impossible for it to satisfy an estimate of the form (6).

- On the other hand, by (25) we can infer that

$$\frac{1}{\pi}\int_0^{2\pi} u(\theta,t)(\cos\theta\ ,\sin\theta)\,d\theta \in \left[-\frac{8\varepsilon}{\pi},\frac{8\varepsilon}{\pi}\right]\times\left[-\frac{8\varepsilon}{\pi},\frac{8\varepsilon}{\pi}\right]$$

for all $t\in[T,\infty)$, which means that the average of position vectors of $\gamma(\cdot,t)$ is even closer to the origin $(0,0)$ for $t\in[T,\infty)$. **Thus from this example we may say that, in general, the concept of the center of expansion is not the same as that of the asymptotic average of position vectors.**

Reference

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