Maximal Exponents of Polyhedral Cones

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Joint work with R. Loewy and M.A. Perles

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• Prologue

$A \geq 0$ (entrywise nonnegative) is **primitive** if there exists $p \in \mathbb{Z}_+$ such that $A^p > 0$ (positive).

Theorem 1. (Wielandt, 1950) *For any* $n \times n$ *primitive* $A, \gamma(A) \leq (n-1)^2 + 1$. *Equality holds* iff

$$P^T A P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

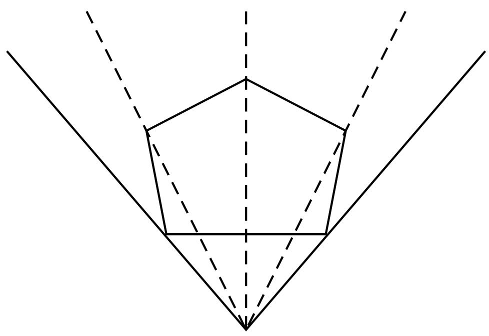
Question (S. Kirkland, 1999) If K is a polyhedral cone in $\mathbb{R}^n$ with m extreme rays, and A is K-primitive, is $(m-1)^2 + 1$ always an upper bound for the exponent $\gamma(A)$ of A (i.e., the least positive integer k such that $A^k(K \setminus \{0\}) \subseteq \text{int}\, K$) ?

• Definitions

K: a **proper cone** in $\mathbb{R}^n$ — closed, pointed, full convex cone; that is, $K + K \subseteq K$; $\alpha K \subseteq K$ for all $\alpha > 0$; $K \cap (-K) = \{0\}$; $\mathrm{span\,K} = \mathrm{K}$; topologically closed, and with nonempty interior w.r.t. usual topology.

Polyhedral cone (finitely generated cone)

$\mathbb{R}^n_+$: **nonnegative orthant**



A polyhedral cone with five extreme rays

A is K**-nonnegative**, $A \geq^K 0$: $AK \subseteq K$.

A is K**-primitive**: A is K-nonnegative and $\exists k$ s.t. $A^k(K \setminus \{0\}) \subseteq \operatorname{int} K$; **exponent of** A, denoted by $\gamma(A)$, for least such k.

$\gamma(K) = \max\{\gamma(A) : A \text{ is } K\text{-primitive}\}$, known as **exponent of** K.

$\gamma(\mathbb{R}^n_+) = (n-1)^2 + 1$.

$\mathcal{P}(m, n)$: set of n-dimensional polyhedral cones with m extreme rays ($3 \leq n \leq m$).

Question. What is $\max\{\gamma(K) : K \in \mathcal{P}(m, n)\}$?

In the study of exponents, the general polyhedral cone case is different from the classical nonnegative matrix case. It is considerably more difficult.

- Background research

Work on exponents of polyhedral cones can be considered as a ramification of the *geometric spectral theory of nonnegative linear operators* — Perron-Frobenius theory of cone-preserving maps.

Joint work with Hans Schneider (and S.-F. Wu): 6 research papers (totally 216 pages), a research-expository paper (71 pages) and an expository article.

B.S.-Tam and H. Schneider, Matrices leaving a cone invariant, in *Handbook of Linear Algebra*, Chapter 26, edited by L. Hogben, Chapman & Hall, 2007.

B.S.-Tam, A cone-theoretic approach to the spectral theory of positive linear operators: the finite-dimensional case, *Taiwanese J. Math.* 5 (2001), 207-277.

- Equivalent problem:

To determine the maximum value of $\gamma(C)$ over all $(n-1)$-polytopes C with m extreme points, having the origin as an interior point, where $\gamma(C)$ is the maximum of the exponents of C-primitive matrices.

When C is a symmetric convex body, not necessarily a polytope, C can be used to define a norm. Then $\gamma(C)$ is equal to the *critical exponent* of the induced norm, that is, the smallest positive integer k with the property that $\|A^k\| = \|A\| = 1$ imply $\|A\|^l = 1$ for all positive integers l. Much studied by V. Pták.

• Major result

Main Theorem.

$$\max\{\gamma(K) : K \in \mathcal{P}(m,n)\}$$
$$= \begin{cases} (n-1)(m-1)+1 & \text{for } m \text{ even, or } m,n \text{ both odd} \\ (n-1)(m-1) & \text{for } m \text{ odd}, n \text{ even} \ . \end{cases}$$

$K_0 \in \mathcal{P}(m,n)$ is an **exp-maximal cone**:
$\gamma(K_0) = \max\{\gamma(K) : K \in \mathcal{P}(m,n)\}$.

A is **exp-maximal K-primitive**: $\gamma(A) = \gamma(K)$, (K, exp-maximal cone)

Uniqueness issue: For optimal cones and optimal K-primitive matrices ?

$A_1 \geq^{K_1} 0$ and $A_2 \geq^{K_2} 0$ are **cone-equivalent** if $\exists$ linear isomorphism P s.t. $PK_2 = K_1$ and $A_2 = P^{-1}A_1P$.

• Further definitions.

For an $n \times n$ matrix A, the **usual digraph** $G(A)$ of A has vertex set $\{1, \ldots, n\}$; (i, j) is an arc if and only if $a_{ij} \neq 0$.

For a K-primitive matrix A,

$\gamma(A, x) := \max\{k : A^k x \in \mathrm{int\,K}\}$, **local exponent**.

Clearly, $\gamma(A) = \max\{\gamma(A, x) : x$ extreme vector of $K\}$.

$\Phi(x) :=$ **face of** K **generated by** x.

$D_K(A)$, **digraph associated with** A:

vertex set consists of extreme rays of K; $(\Phi(x), \Phi(y))$ is an arc iff $\Phi(y) \subseteq \Phi(Ax)$.

When $A \geq 0$, $D_{\mathbb{R}^n_+}(A) \cong$ usual digraph of A^T.

- Relevance of circuits of $D_K(A)$

$\exists$ a directed walk of length k from $\Phi(x)$ to $\Phi(y)$ implies $\quad \Phi(y) \subseteq \Phi(A^k x)$

If there exists a circuit of length k containing $\Phi(x)$ then we have $\Phi(x) \subseteq \Phi(A^k x)$.

If $\Phi(x) = \Phi(A^k x)$ then A^k is K-reducible, contradicting K-primitivity of A. So we have $\Phi(x) \subset \Phi(A^k x)$.

Continuing the argument, we obtain

$\Phi(x) \subset \Phi(A^k x) \subset \Phi(A^{2k} x) \subset \cdots \Phi(A^{lk} x) = K$, implying $A^{lk} x \in \operatorname{int} \mathrm{K}$.

- Special digraphs for K-primitive matrices

The longer the shortest circuit in $D_K(A)$, the more likely $\gamma(A)$ has a larger value.

Lemma 1. *Let $K \in \mathcal{P}(m,n)$ $(3 \le n \le m)$ and let A be a K-primitive matrix. Then the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ equals $m-1$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure* 1 *or Figure* 2*.*

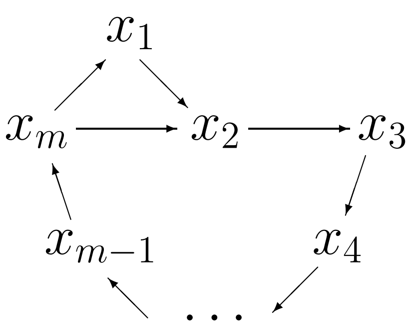


Figure 1.

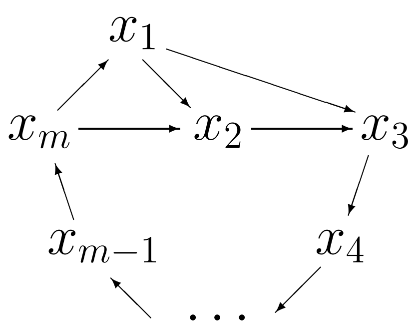


Figure 2.

$D_K(A)$ given by Figure 1 implies: $Ax_i = \lambda_i x_{i+1}$ for $i = 1, \dots, m-1$ and $Ax_m = a_1 x_1 + a_2 x_2$, where λ_i's and $a_1, a_2 > 0$.

- A preliminary result

Theorem 2. *Let $K \in \mathcal{P}(m, n)$.*

(i) *If A is K-primitive, then $\gamma(A) \leq (m_A - 1)(m - 1) + 1$, where m_A is the degree of the minimal polynomial of A.*

(ii) $\gamma(K) \leq (n-1)(m-1)+1$, *equality holds only if $\exists$ K-primitive matrix A s.t. $D_K(A)$, the digraph associated with A, is (up to graph isomorphism) given by Fig. 1.*

(iii) *If A is K-primitive and $\gamma(A) = (n-1)(m-1)$ then $D_K(A)$ is given by Fig. 1 or Fig. 2.*

Kirkland's question: Always $\gamma(A) \leq (m-1)^2 + 1$?
Yes !

- When $D_K(A)$ is given by Fig. 1 or Fig. 2:

Minimal cone: n-dim. polyhedral cone with $n+1$ extreme rays.

Lemma 2. *Let $K \in \mathcal{P}(m,n)$. Let $A \geq^K 0$. Suppose $D_K(A)$ given by Fig. 1 or Fig. 2.*

(i) *If K is non-simplicial then K is indecomposable or K is an even-dimensional minimal cone which is the direct sum of a ray and an indecomp. minimal cone with a balanced relation for its extreme vectors.*

(ii) *$\gamma(A) = \gamma(A, x_1)$ or $\gamma(A, x_2)$, depending on whether $D_K(A)$ given by Fig. 1 or Fig. 2.*

(iii) *A is K-primitive, nonsingular, non-derogatory, and has a unique annihilating polynomial of the form $t^m - at - b$, where $a, b > 0$.*

• The minimal cone case:

Theorem 3. *Let $n \geq 3$ be a given positive integer.*

(i)

$$\max\{\gamma(K) : K \in \mathcal{P}(n+1, n)\} = \begin{cases} n^2 - n + 1 & \textit{for odd } n \\ n^2 - n & \textit{for even } n. \end{cases}$$

(ii) *Let $K \in \mathcal{P}(n+1, n)$.*

For odd n, K is exp-max. iff K is indecomp. with balanced relation for its extreme vectors.

For even n, K is exp-max. iff K is indecomp. with a balanced relation for extreme vectors, or K = a ray $\oplus$ an indecomp. minimal cone with a balanced relation for extreme vectors.

(iii) *Concerning exp-maximal K-primitive matrices.*

- The 3-dimensional case:

Theorem 4. *Let* $m \geq 3$.

(i) $\max\{\gamma(K) : K \in \mathcal{P}(m,3)\} = 2m - 1$.

(ii) $\forall K \in \mathcal{P}(m,3)$, *K is exp-maximal iff* $\exists$ *K-primitive A s.t.* $D_K(A)$ *is given by Fig.* 1.

(iii) $\cdots$

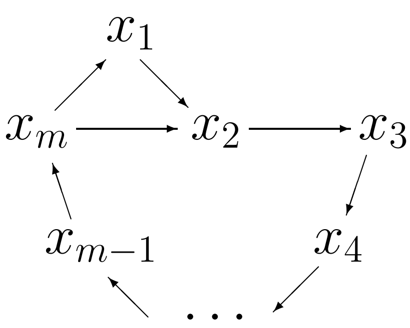


Figure 1.

If $\Phi(x_i)$ and $\Phi(x_{i+1})$ are neighborly extreme rays of K for $i = 1, \ldots, m$ and A satisfies

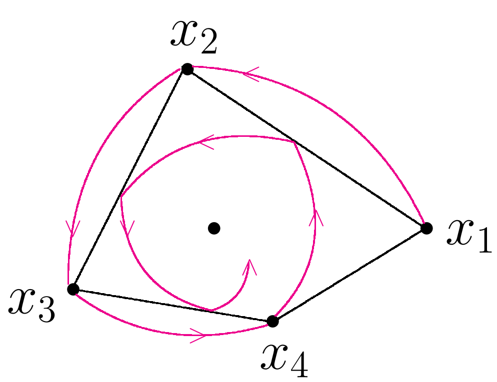
$$\begin{aligned} Ax_i &= x_{i+1} \text{ for } i = 1, \ldots, m-1 \\ \text{and } Ax_m &= (1-c)x_1 + cx_2,\ 0 < c < 1, \end{aligned}$$

then $D_K(A)$ is given by Figure 1.

Try: $x_1 = (1, 0, 1)^T, x_2 = (r\cos\theta, r\sin\theta, 1)^T, \ldots$
$x_m = (r^{m-1}\cos(m-1)\theta, r^{m-1}\sin(m-1)\theta, 1)^T$

$$\text{and} \quad A = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \oplus [1].$$

$m = 4$:



Work on the minimal cone case and the 3-dimensional case has led us to conjecture the maximum value of $\gamma(K)$ as given in our main result.

Theorem 3 continued:

(iii) $\forall\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, let r_θ denote the unique positive real root of the poly. $g_\theta(t)$ given by :

$$g_\theta(t) = \frac{\sin(m-1)\theta}{\sin\theta} t^m - \frac{\sin m\theta}{\sin\theta} t^{m-1} + 1.$$

Let K_θ be the polyhedral cone in $\mathbb{R}^3$ generated by the vectors

$$x_j(\theta) := \begin{bmatrix} r_\theta^{j-1}\cos(j-1)\theta \\ r_\theta^{j-1}\sin(j-1)\theta \\ 1 \end{bmatrix}, \; j = 1, \ldots, m.$$

Also, let $A_\theta = r_\theta \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \oplus [1]$. Then $x_1(\theta), \ldots, x_m(\theta)$ are the extreme vectors of K_θ, K_θ is an exp-maximal polyhedral cone and A_θ is an exp-maximal K_θ-primitive matrix.

• The higher-dimensional case:

Lemma 3. *For any positive integers* $m, n, 3 \le n \le m$*, there exists* $K \in \mathcal{P}(m,n)$ *with a* K*-primitive* A *such that* $D_K(A)$ *given by Fig.* 1*.*

For even m, the desired exp-maximal cone K has extreme vectors $x_1, \ldots, x_m$ given by:

$$x_j = \begin{bmatrix} r_1^{j-1}\cos(j-1)\theta_1 \\ r_1^{j-1}\sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1}\cos(j-1)\theta_p \\ r_p^{j-1}\sin(j-1)\theta_p \\ a^{j-1} \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} r_1^{j-1}\cos(j-1)\theta_1 \\ r_1^{j-1}\sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1}\cos(j-1)\theta_p \\ r_p^{j-1}\sin(j-1)\theta_p \\ 1 \end{bmatrix},$$

depending on whether n even or odd. Here $1, a$ are real roots and $r_j e^{\pm\theta_j}$ ($j = 1, \ldots, \frac{m-2}{2}$) are non-real complex roots of

$$h(t) = t^m - ct - (1-c),$$

where $c \in (0,1)$, suitably chosen; also, $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$.

The desired exp-max. K-primitive matrix A is:

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [a] \oplus [1]$$

or

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [1],$$

depending on whether n is even or odd.

Main Theorem.

$$\max\{\gamma(K) : K \in \mathcal{P}(m,n)\}$$
$$= \begin{cases} (n-1)(m-1)+1 & \text{for } m \text{ even, or } m,n \text{ both odd} \\ (n-1)(m-1) & \text{for } m \text{ odd}, n \text{ even}. \end{cases}$$

Already have

$$\max\{\gamma(K) : K \in \mathcal{P}(m,n)\} \leq (n-1)(m-1)+1.$$

By calculation,

$$\Phi(A^{(n-1)(m-1)-1}x_1) = \Phi(x_{m-n+1} + \cdots + x_{m-1}),$$

$$\Phi(A^{(n-1)(m-1)}x_1) = \Phi(x_{m-n+2} + x_{m-n+3} + \cdots + x_m).$$

Remains to show

(1) When n even, m odd: $\forall K \in \mathcal{P}(m,n)$, $\gamma(K) \leq (n-1)(m-1)$.

Loewy and I couldn't prove it for a couple of years.

(2) For m even or m and n both odd (resp., for m odd and n even), $\exists K \in \mathcal{P}(m,n)$ and K-primitive matrix A s.t. $D_K(A)$ is given by Fig. 1 and $x_{m-n+2} + x_{m-n+3} + \cdots + x_m \in \partial K$ (resp., $x_{m-n+1} + x_{m-n+2} + \cdots + x_{m-1} \in \partial K$).

To prove (2) when m is even or m, n are both odd: As $c \to 0^+$, $h(t) := t^m - ct - (1-c) \to t^m - 1$. In the proof of Lemma 3, take $r_j e^{\pm i\theta_j} \approx e^{\pm \frac{2\pi j}{m} i}$. Then $K \approx K_0, x_j \approx y_j, j = 1, \ldots, m$, where $K_0 =$

pos$\{\mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m}\}$,

$$y_j = \begin{bmatrix} \cos(j-1)\frac{2\pi}{m} \\ \sin(j-1)\frac{2\pi}{m} \\ \cos(j-1)\frac{4\pi}{m} \\ \sin(j-1)\frac{4\pi}{m} \\ \vdots \\ \cos(j-1)\frac{(n-1)\pi}{m} \\ \sin(j-1)\frac{(n-1)\pi}{m} \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} \cos(j-1)\frac{2\pi}{m} \\ \sin(j-1)\frac{2\pi}{m} \\ \cos(j-1)\frac{4\pi}{m} \\ \sin(j-1)\frac{4\pi}{m} \\ \vdots \\ \cos(j-1)\frac{(n-2)\pi}{m} \\ \sin(j-1)\frac{(n-2)\pi}{m} \\ (-1)^{j-1} \\ 1 \end{bmatrix},$$

depending n odd, or n, m both even. Reduces to showing that $\sum_{j=1}^{n-1} y_j \in \partial K_0$ and generates a simplicial face.

When m is odd and n is even, still use the K in the proof of Lemma 3, but letting $c \to 1^-$ instead.

The problem is reduced to showing that $\det Q_p$ ($p = n, \ldots, m$) are nonzero and have the same sign, where

$$Q_p := \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_p \end{bmatrix}.$$

After hard work we succeeded in finding a proof. It involves certain generalized Vandermonde matrices, the complete symmetric polynomials, the Jacobi-Trudi determiant, and a nontrivial result about polynomials with nonnegative ceofficients.

We (Loewy and I) published two papers and was about to submit the third paper:

1. Maximal exponents of polyhedral cones (I), J. Math. Anal. Appl. 365 (2010), 570-583.

2. Maximal exponents of polyhedral cones (II), Linear Algebra Appl. 432 (2010), 2861-2878.

The conclusion of a story is often changed by the happening of another story.

S. Sergeev $\xrightarrow{\text{Dec.2009,visit}}$ H. Schneider $\longrightarrow$ V.S. Grinberg (1989 paper) $\longrightarrow$ M.A. Perles (1964 Ph.D. thesis):

V.S. Grinberg, Wielandt-type bounds for primitive matrices of partially ordered sets, Mat. Zametki 45 (1989), 30-35 (Russian); translation in Math. Notes 45 (1989), 450-454.

M.A. Perles, *Critical Exponents of Convex Bodies*, Ph.D. thesis, 1964, 169 pages (in Hebrew).

M.A. Perles, Critical exponents of convex sets, in: Proceedings of the Colloquium in Convexity (Copenhagen, 1965), (1967), 221-228.

(1) and (2) are done in Perles' Ph.D. thesis, unpublished.

- Perles' proof of (2):

He makes use of the following

Lemma 4. (*Scott, 1879*) *Let p be a given positive integer. For any $\theta \in \mathbb{R}$, denote by $x(\theta)$ the vector*

$$(\cos\theta, \sin\theta, \cos 2\,\theta, \sin 2\,\theta, \cdots, \cos p\,\theta, \sin p\,\theta)^T$$

of $\mathbb{R}^{2p}$. For any $\theta_0, \theta_1, \ldots, \theta_{2p} \in \mathbb{R}$, we have:

$$\begin{vmatrix} x(\theta_0) & x(\theta_1) & \cdots & x(\theta_{2p}) \\ 1 & 1 & \cdots & 1 \end{vmatrix} = 4^{p^2} \prod_{0 \le i < j \le 2p} \sin \frac{1}{2}(\theta_j - \theta_i).$$

• Perles' proof of (1): If $D_K(A)$ is given by Figure 1 and $\gamma(A) = (n-1)(m-1)+1$, then:

$$
\begin{aligned}
& \mathrm{sgn}|\mathrm{A}|\mathrm{sgn}\left|x_m \;\; x_1 \;\; \cdots \;\; x_{n-1}\right| \\
= \;& \mathrm{sgn}\left|Ax_m \;\; Ax_1 \;\; \cdots \;\; Ax_{n-1}\right| \\
= \;& \mathrm{sgn}\left|(1-c)x_1 + cx_2 \;\; x_2 \;\; \cdots \;\; x_n\right| \\
= \;& \mathrm{sgn}\left|x_1 \;\; x_2 \;\; \cdots \;\; x_n\right| \\
= \;& (-1)^{n-1}\mathrm{sgn}\left|x_n \;\; x_1 \;\; \cdots \;\; x_{n-1}\right| \neq 0.
\end{aligned}
$$

We obtain $\mathrm{sgn}(|\mathrm{A}|) = (-1)^{\mathrm{n}-1}$. Similarly, we have

$$
\begin{aligned}
& (\mathrm{sgn}(|\mathrm{A}|))^{\mathrm{m}}\mathrm{sgn}(\left|x_{m-n+2} \;\; \cdots \;\; x_m \;\; x_1\right|) \\
= \;& \mathrm{sgn}\left|x_{m-n+2} \;\; \cdots \;\; x_m \;\; \alpha x_1 + \beta x_2\right|,
\end{aligned}
$$

and hence $(\mathrm{sgn}(|\mathrm{A}|))^{\mathrm{m}} = 1$. But $\mathrm{sgn}(|\mathrm{A}|) = (-1)^{\mathrm{n}-1}$, so $(-1)^{(n-1)m} = 1$ or $(n-1)m \equiv 0 \pmod{2}$.

• Exp-maximal cones and exp-maximal primitive matrices

Theorem 5. (a) *For every positive integer* $m \geq 5$*, up to linear isomorphism,* K_θ *are precisely all the exp-maximal cones in* $\mathcal{P}(m, 3)$*, uncountably infinitely many of them.*

(b) *When* $m \geq 6$*, we have:*

(1) *For each* $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$*, there is* (*up to multiples*) *only one exp-maximal* K_θ*-primitive matrix.*

(2) *The automorphism group of* K_θ *consists of scalar matrices only.*

(3) *For any* $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1}), \theta_1 \neq \theta_2$*, the cones* $K_{\theta_1}, K_{\theta_2}$ *are not linearly isomorphic.*

(c) *When* $m = 5$*,*

(i) *The automorphism group of* K_θ *consists of the identity matrix and an involution* P, *different from the identity matrix, together with their positive multiples.*

(ii) *For each* $\theta \in (\frac{2\pi}{5}, \frac{\pi}{2})$, *there are (up to multiples) precisely two exp-maximal* K_θ*-primitive, namely,* K_θ *and* $P^{-1}A_\theta P$.

(iii) *For any* $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1}), \theta_1 \neq \theta_2$, *the cones* $K_{\theta_1}, K_{\theta_2}$ *are not linearly isomorphic.*

To show that when $m \geq 6$, (up to multiples) exp-maximal K_θ-primitive matrix is unique:

$A_\theta x_j = x_{j+1}$ for $j = 1, \ldots m-1$.

Suppose B is another exp-maximal K_θ-primitive matrix. Then there exists $p, 1 \leq p \leq m$ such that $Bx_j = \lambda_j x_{j+1}$ for $j = p, p+1, \ldots, p+m-2$, where $\lambda_j > 0$ (or ...).

Hence Bx_j is a positive multiple of $A_\theta x_j$ for $m-2$ or at least 4 x_j's (as $m \geq 6$). As the underlying space is 3-dimensional, B is a positive multiple of A_θ.