Introduction to Semidefinite Programming

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Abstract

- The main purpose of this lecture is an introduction of semidefinite programs for graduate students and researchers who are not familiar to this subject and/or who want to look over SDPs quickly.
- Assuming the basics of linear programs and linear algebra, the lecture places the main emphasis on the basic theory of SDPs.
- Some examples and applications of SDPs are also presented to show the significance of SDPs in the field of optimization.

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SDP is an extension of LP to the space of symmetric matrices.

LP: minimize $-X_{11} - 2X_{12} - 5X_{22}$
subject to $2X_{11} + 3X_{12} + X_{22} = 7,\ X_{11} + X_{12} \geq 1,$
$X_{11} \geq 0,\ X_{12} \geq 0,\ X_{22} \geq 0.$

SDP: minimize $-X_{11} - 2X_{12} - 5X_{22}$
subject to $2X_{11} + 3X_{12} + X_{22} = 7,\ X_{11} + X_{12} \geq 1,$
$X_{11} \geq 0,\ X_{12} \geq 0,\ X_{22} \geq 0,$
$\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \boldsymbol{O}$ (positive semidefinite).

- Both LP and SDP have linear objective functions in real variables $X_{11},\ X_{12},\ X_{22}$.
- Both LP and SDP have linear equality and inequality constraints in real variables $X_{11},\ X_{12},\ X_{22}$.

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$\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \boldsymbol{O}$ (positive semidefinite).

- SDP has a psd constraint in $\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$, or
$X_{11} \geq 0,\ X_{22} \geq 0,\ X_{11}X_{22} - X_{12}^2 \geq 0,$ which requires $X_{11},\ X_{12},\ X_{22}$ 'dependent nonlinearly', while $X_{11} \geq 0,\ X_{12} \geq 0,\ X_{22} \geq 0$ in LP are linear and separable.

SDP is an extension of LP to the space of symmetric matrices.

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$\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \boldsymbol{O}$ (positive semidefinite).

- The feasible region of LP and the feasible region of SDP are convex sets, but the former is polyhedral while the latter is non-polyhedral.

Exercise.
Draw a picture of the set $\{(X_{11}, X_{12}, X_{22}) : \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \boldsymbol{O}\}$.

Contents

Lots of Applications to Various Problems

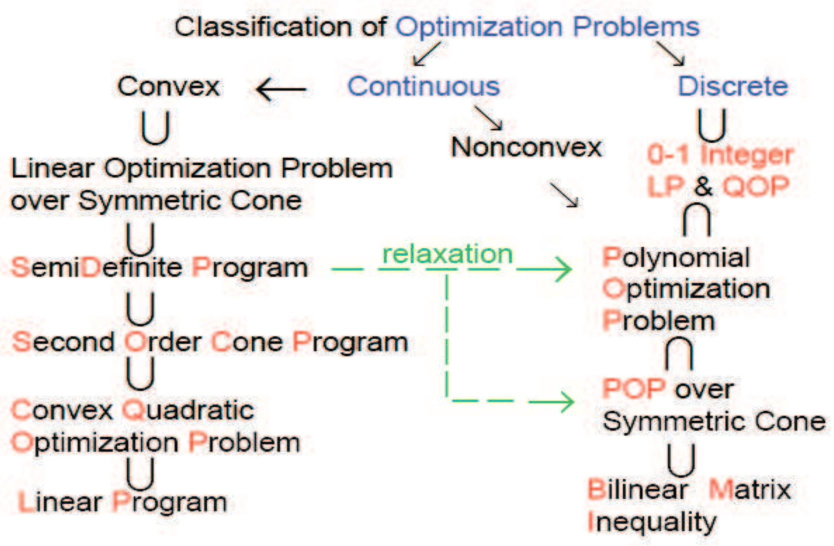
- Systems and control theory — Linear Matrix Inequality [6]
- SDP relaxations of combinatorial and nonconvex problems
  - Max cut and max clique problems [14]
  - 0-1 integer linear programs [24]
  - Polynomial optimization problems [22, 35]
- Robust optimization [4]
- Quantum chemistry [51]
- Moment problems (applied probability) [5, 23]
- . . .

Survey articles — Todd [39] ,Vandenberghe-Boyd [45]
Handbook of SDP — Wolkowicz-Saigal-Vandenberghe [46]
Web pages — Helmberg[15], Wolkowicz [47]

Theory

- Self-concordant theory [33]
- Euclidean Jordan algebra [10, 36]
- Polynomial-time primal-dual interior-point methods [1, 17, 20, 27, 34]

SDP serves as a core convex optimization problem



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1. LP versus SDP
2. Why is SDP interesting and important?
3. The equality standard form
4. Some basic properties on positive semidefinite matrices and their inner product
5. General SDPs
6. Some examples
7. Duality
8. The central trajectory
9. Numerical methods for SDPs
10. Numerical results

(LP) minimize $\boldsymbol{a}_0 \cdot \boldsymbol{x}$
subject to $\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (1 \leq p \leq m), \ \mathbb{R}^n \ni \boldsymbol{x} \geq \mathbf{0}$.

Here
$\mathbb{R}$: the set (linear space) of real numbers,
$\mathbb{R}^n$: the linear space of n dim. vectors,
$\boldsymbol{a}_p \in \mathbb{R}^n$: data, n dim. vector $(1 \leq p \leq m)$,
$b_p \in \mathbb{R}$: data, real number $(1 \leq p \leq m)$,
$\boldsymbol{x} \in \mathbb{R}^n$: variable, n dim. vector,
$\boldsymbol{a}_p \cdot \boldsymbol{x}$ = $\sum_{i=1}^n [\boldsymbol{a}_p]_i \boldsymbol{x}_i$ (the inner product of $\boldsymbol{a}_p$ and $\boldsymbol{x}$).

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(SDP) minimize $\boldsymbol{A}_0 \bullet \boldsymbol{X}$
subject to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m), \ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$.

$\mathbb{S}^n$: the linear space of $n \times n$ symmetric matrices,
$\boldsymbol{A}_p \in \mathbb{S}^n$: data, $n \times n$ symmetric matrix $(0 \leq p \leq m)$,
$b_p \in \mathbb{R}$: data, real number $(1 \leq p \leq m)$,
$\boldsymbol{X} \in \mathbb{S}^n$: $n \times n$ variable, symmetric matrix;

$$\boldsymbol{X} = (X_{ij}) = \begin{pmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \dots & X_{nn} \end{pmatrix} \in \mathbb{S}^n,$$

$X_{ij} = X_{ji} \in \mathbb{R} \ (1 \leq i \leq j \leq n)$,

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$\boldsymbol{X} \in \mathbb{S}^n_+ \ \Leftrightarrow \ \boldsymbol{X} \in \mathbb{S}^n$ is positive semidefinite,
$\boldsymbol{X} \succeq \boldsymbol{O} \ \Leftrightarrow \ \boldsymbol{X} \in \mathbb{S}^n_+$ for some n,
$\boldsymbol{A}_p \bullet \boldsymbol{X} \ = \ \sum_{i=1}^n \sum_{j=1}^n [\boldsymbol{A}_p]_{ij} \boldsymbol{X}_{ij}$
(the inner product of $\boldsymbol{A}_p$ and $\boldsymbol{X}$).

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subject to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m), \ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$.

$$\Uparrow \quad \left\{ \begin{array}{l} m = 2, \ n = 2, \ b_1 = 7, \ b_2 = 9, \\ \boldsymbol{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \ \boldsymbol{A}_0 = \begin{pmatrix} -1 & -1 \\ -1 & -5 \end{pmatrix}, \\ \boldsymbol{A}_1 = \begin{pmatrix} 2 & 1.5 \\ 1.5 & 1 \end{pmatrix}, \ \boldsymbol{A}_2 = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 3 \end{pmatrix}. \end{array} \right.$$

minimize $-X_{11} - 2X_{12} - 5X_{22}$
subject to $2X_{11} + 3X_{12} + X_{22} = 7, \ 2X_{11} + X_{12} + 3X_{22} = 9,$
$$\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \boldsymbol{O}.$$

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(SDP) minimize $\boldsymbol{A}_0 \bullet \boldsymbol{X}$
subject to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m), \ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$.

$\mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$: semidefinite constraint.

- Definition: $\boldsymbol{X} \succeq \boldsymbol{O}$ if $\boldsymbol{u}^T\boldsymbol{X}\boldsymbol{u} = \sum_{i=1}^n \sum_{j=1}^n X_{ij}u_iu_j \geq 0$ for $\forall \boldsymbol{u} \in \mathbb{R}^n$.
- Definition: $\boldsymbol{X} \succ \boldsymbol{O}$ if $\boldsymbol{u}^T\boldsymbol{X}\boldsymbol{u} = \sum_{i=1}^n \sum_{j=1}^n X_{ij}u_iu_j > 0$ for $\forall \boldsymbol{u} \neq \mathbf{0}$.
- $\boldsymbol{X} \in \mathbb{S}^n \Rightarrow$ all n e.values are real.
- $\boldsymbol{X} \succeq \boldsymbol{O}$ ($\succ \boldsymbol{O}$) $\Leftrightarrow$ all n e.values ≥ 0 (> 0).
- $\boldsymbol{X} \succeq \boldsymbol{O}$ ($\succ \boldsymbol{O}$) $\Leftrightarrow$ all principal minors ≥ 0 (> 0).
- $\boldsymbol{X} \succeq \boldsymbol{O}$ ($\succ \boldsymbol{O}$) $\Rightarrow$ all diagonal X_{ii}'s ≥ 0 (> 0).
- $\boldsymbol{X} \succeq \boldsymbol{O}$ and $X_{ii} = 0 \Rightarrow X_{ij} = 0 \ (\forall j)$.

(SDP) minimize $\boldsymbol{A}_0 \bullet \boldsymbol{X}$
subject to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m), \ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$.

$\mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$: semidefinite constraint.

- $\boldsymbol{X} \succeq \boldsymbol{O}$ ($\succ \boldsymbol{O}$) $\Leftrightarrow \exists n \times n$ (nonsingular) $\boldsymbol{B}$; $\boldsymbol{X} = \boldsymbol{B}\boldsymbol{B}^T$ (factorization).
- $\boldsymbol{X} \succeq \boldsymbol{O} \Leftrightarrow \exists n \times n$ lower triang. $\boldsymbol{L}$; $\boldsymbol{X} = \boldsymbol{L}\boldsymbol{L}^T$ (Cholesky factorization).
- $\boldsymbol{X} \succeq \boldsymbol{O} \Leftrightarrow \exists n \times n$ orthogonal P and $\exists n \times n$ diagonal $\boldsymbol{D}$;
  $\boldsymbol{X} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^T$ (orthogonal decompostion).
  Here each diagonal element $\lambda_i = D_{ii}$ of D is an eigenvalue of $\boldsymbol{X}$ and each ith column p_i of P an eigenvector corresponding to λ_i.
- $\boldsymbol{X} \succeq \boldsymbol{O} \Leftrightarrow \exists \boldsymbol{C} \in \mathbb{S}^n_+$; $\boldsymbol{X} = \boldsymbol{C}^2 \Leftarrow$ Take $\boldsymbol{C} = \boldsymbol{P}(\boldsymbol{D})^{1/2}\boldsymbol{P}^T$;
  $$\boldsymbol{C}^2 = \left(\boldsymbol{P}(\boldsymbol{D})^{1/2}\boldsymbol{P}^T\right)\left(\boldsymbol{P}(\boldsymbol{D})^{1/2}\boldsymbol{P}^T\right) = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^T = \boldsymbol{X}.$$
  We will write $\boldsymbol{X} = \left(\sqrt{X}\right)^2$.

(SDP) minimize $\boldsymbol{A}_0 \bullet \boldsymbol{X}$
subject to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m), \ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$.

$\mathbb{S}^n$: a linear space with dimension $n(n+1)/2$.

- $\boldsymbol{X} + \boldsymbol{Y} \in \mathbb{S}^n$ for $\forall \boldsymbol{X} \in \mathbb{S}^n$ and $\forall \boldsymbol{Y} \in \mathbb{S}^n$.
- $\alpha \boldsymbol{X} \in \mathbb{S}^n$ for $\forall \alpha \in \mathbb{R}$ and $\forall \boldsymbol{X} \in \mathbb{S}^n$.
- linear independence.
- a basis consisting of $n(n+1)/2$.

Example. $n = 2$. Note that $X_{12} = X_{21}$.

$$2\begin{pmatrix} 1.1 & -0.5 \\ -0.5 & 2.4 \end{pmatrix} + 0.5\begin{pmatrix} 2.4 & 0.6 \\ 0.6 & 1.2 \end{pmatrix} = \begin{pmatrix} 3.4 & 0.7 \\ 0.7 & 5.4 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ : \text{ a basis of } \mathbb{S}^2.$$

(SDP) minimize $\boldsymbol{A}_0 \bullet \boldsymbol{X}$
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- linear independence.
- a basis consisting of $n(n+1)/2$.
- For every $\boldsymbol{A},\ \boldsymbol{X} \in \mathbb{S}^n$, the inner product $\boldsymbol{A} \bullet \boldsymbol{X}$ is defined;

$$\begin{aligned} \boldsymbol{A} \bullet \boldsymbol{X} &= \sum_{i=1}^n \left(\sum_{j=1}^n A_{ij} X_{ij} \right) \\ &= \sum_{i=1}^n \underbrace{\left(\sum_{j=1}^n A_{ij} X_{ji} \right)}_{(i,i)\text{th element of } \boldsymbol{A}\boldsymbol{X}} = \text{trace } \boldsymbol{A}\boldsymbol{X}. \end{aligned}$$

- $\boldsymbol{u}^T \boldsymbol{X} \boldsymbol{u} = \text{trace } \boldsymbol{u}^T \boldsymbol{X} \boldsymbol{u} = \text{trace } \boldsymbol{X} \boldsymbol{u} \boldsymbol{u}^T = \boldsymbol{X} \bullet \boldsymbol{u} \boldsymbol{u}^T$

(SDP) minimize $\boldsymbol{A}_0 \bullet \boldsymbol{X}$
subject to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \le p \le m),\ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$.

$\mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$ and the inner product $\boldsymbol{X} \bullet \boldsymbol{Y}$.

- $\mathbb{S}^n_+ \subseteq (\mathbb{S}^n_+)^* \equiv \left\{ \boldsymbol{Y} \in \mathbb{S}^n : \boldsymbol{Y} \bullet \boldsymbol{X} \ge 0 \text{ for } \forall \boldsymbol{X} \in \mathbb{S}^n_+ \right\}$.
- $\mathbb{S}^n_+ \supseteq (\mathbb{S}^n_+)^*$. Hence $\mathbb{S}^n_+ = (\mathbb{S}^n_+)^*$ (self-dual).

.

(SDP) minimize $\boldsymbol{A}_0 \bullet \boldsymbol{X}$
subject to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \le p \le m),\ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}$.

(SDP) minimize $A_0 \bullet X$

subject to $A_p \bullet X = b_p \ (1 \le p \le m), \ \mathbb{S}^n \ni X \succeq O$.

Common properties on

$$\mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n : x \ge 0\}, \ \mathbb{S}^n_+ \equiv \{X \in \mathbb{S}^n : X \succeq O\}.$$

- $\mathbb{R}^n_+$ is a cone; $\alpha X \in \mathbb{R}^n_+$ if $\alpha \ge 0$, $x \in \mathbb{R}^n_+$.
- $\mathbb{R}^n_+$ is convex;
  $\lambda x + (1-\lambda) x \in \mathbb{R}^n_+$ if $0 \le \lambda \le 1, \ x, \ y \in \mathbb{R}^n_+$.
- self-dual;
  $(\mathbb{R}^n_+)^* \equiv \left\{y \in \mathbb{R}^n : y \bullet x \ge 0 \text{ for } \forall x \in \mathbb{R}^n_+\right\} = \mathbb{R}^n_+$.
- $x, \ y \in \mathbb{R}^n_+$ and $x \cdot y = 0 \Longrightarrow x_i y_i = 0 \ (1 \le i \le n)$.

- $\mathbb{S}^n_+$ is a cone; $\alpha X \in \mathbb{S}^n_+$ if $\alpha \ge 0$ and $X \in \mathbb{S}^n_+$.
- $\mathbb{S}^n_+$ is convex;
  $\lambda X + (1-\lambda) Y \in \mathbb{S}^n_+$ if $0 \le \lambda \le 1$ and $X, \ Y \in \mathbb{S}^n_+$.
- self-dual;
  $(\mathbb{S}^n_+)^* \equiv \left\{Y \in \mathbb{S}^n : Y \bullet X \ge 0 \text{ for } \forall X \in \mathbb{S}^n_+\right\} = \mathbb{S}^n_+$.
- $X, \ Y \in \mathbb{S}^n_+$ and $X \bullet Y = 0 \Longrightarrow XY = O$.

Contents

Equality standard form (SDP):

min. $A_0 \bullet X$

sub.to $A_p \bullet X = b_p \ (1 \le p \le m), \ \mathbb{S}^n \ni X \succeq O$.

Equality standard form with multiple matrix variables (SDP)' :

min. $\sum_{q=1}^t A_0^q \bullet X^q$

sub.to $\sum_{q=1}^t A_p^q \bullet X^q = b_p \ (1 \le p \le m)$,
$\mathbb{S}^{n^q} \ni X^q \succeq O \ (1 \le q \le t)$.

- If $n^q = 1 \ (1 \le q \le t)$, (SDP)' is equivalent to the equality standard form of LP, where $A_p^q \in \mathbb{R}$ and $X^q \in \mathbb{R}$.
- Can we transform the above (SDP)' (or the equality standard form of LP) into Equality standard form (SDP)?

Equality standard form (SDP):

min. $A_0 \bullet X$

sub.to $A_p \bullet X = b_p \ (1 \le p \le m), \ \mathbb{S}^n \ni X \succeq O$.

$$\Uparrow \quad n = \sum_{q=1}^t n^q, \ A_p \equiv \begin{pmatrix} A_p^1 & O & \cdots & O \\ O & A_p^2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_p^t \end{pmatrix}.$$

Equality standard form with multiple matrix variables (SDP)' :

min. $\sum_{q=1}^t A_0^q \bullet X^q$

sub.to $\sum_{q=1}^t A_p^q \bullet X^q = b_p \ (1 \le p \le m)$,
$\mathbb{S}^{n^q} \ni X^q \succeq O \ (1 \le q \le t)$.

- If $n^q = 1 \ (1 \le q \le t)$, (SDP)' is equivalent to the equality standard form of LP, where $A_p^q \in \mathbb{R}$ and $X^q \in \mathbb{R}$.
- Can we transform the above (SDP)' (or the equality standard form of LP) into Equality standard form (SDP)?

Exercise. Prove (SDP)' is equivalent to (SDP). Hint: Construct an optimal solution of (SDP) from any optimal solution of (SDP)', and vice versa.

Why do we need a standard from SDP?

(a) A unified SDP model for theory and method of SDPs.

(b) SDP software packages are available only for some standard forms.

Equality standard form (SDP):

min. $\boldsymbol{A}_0 \bullet \boldsymbol{X}$

sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \le p \le m), \ \mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}.$

$\Uparrow \ ?$

An SDP from systems and control theory (SDP)':

min λ

sub. to $\begin{pmatrix} \boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{X} + \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{X}\boldsymbol{B} + \boldsymbol{C}^T\boldsymbol{D} \\ \boldsymbol{B}^T\boldsymbol{X} + \boldsymbol{D}^T\boldsymbol{C} & \boldsymbol{D}^T\boldsymbol{D} - \boldsymbol{I} \end{pmatrix} \preceq \lambda \boldsymbol{I},$

$\boldsymbol{X} \succeq -\lambda \boldsymbol{I}.$

Here $\boldsymbol{X} \in \mathbb{S}^n$ and $\lambda \in \mathbb{R}$ are variables, and $\boldsymbol{A}, \ \boldsymbol{B}, \ \boldsymbol{C}, \ \boldsymbol{D}$ are given data matrices.

- Can we transform the (SDP)' into Equality standard form (SDP)?
- "Yes" in theory, but not practical at all.
- Transform (SDP)' into an LMI standard form (with equality constraints), which corresponds to the dual of an equality standard form with free variables.

A general SDP:

min. a linear function in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q \ (1 \le q \le t)$

sub.to linear equalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,

linear inequalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,

linear (matrix) inequalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,

$x_1, \ldots, x_k \in \mathbb{R}$ (free real variables),

$\boldsymbol{X}^q \succeq \boldsymbol{O} \ (1 \le q \le t)$ (psd matrix variables).

Here a nonnegative x_i is regarded as a 1×1 psd matrix var., and a matrix variable $\boldsymbol{U} \in \mathbb{R}^{k \times m}$ a set of free variables U_{ij}s.

Any real-valued linear function in $\boldsymbol{X} \in \mathbb{S}^n$ can be written as $\boldsymbol{A} \bullet \boldsymbol{X} = \sum_{i=1}^n \sum_{j=1}^n \boldsymbol{A}_{ij} \boldsymbol{X}_{ij}$ for $\exists \boldsymbol{A} \in \mathbb{S}^n$.

- We can transform 'any SDP' into Equality standard form. But such a transformation is neither trivial nor practical in many cases.
- It is easier to reduce an SDP to 'an LMI standard form with equality constraints' than to Equality standard form.

A general SDP:

min. a linear function in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q \ (1 \le q \le t)$

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$\boldsymbol{X}^q \succeq \boldsymbol{O} \ (1 \le q \le t)$ (psd matrix variables).

A general SDP:

min.	a linear function in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$ $(1 \le q \le t)$
sub.to	linear equalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,
	linear inequalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,
	linear (matrix) inequalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,
	$x_1, \ldots, x_k \in \mathbb{R}$ (free real variables),
	$\boldsymbol{X}^q \succeq \boldsymbol{O}$ $(1 \le q \le t)$ (psd matrix variables).

Reduction to 'an LMI standard form with equality constraints'.

Represent each symmetric variable $\boldsymbol{X}^q \in \mathbb{S}^{n^q}$ as a linear combination of a basis $\boldsymbol{E}^q_{ij}$ $(1 \le i \le j \le n^q)$ such that

$$\boldsymbol{X}^q = \sum_{1 \le i \le j \le n^q} \boldsymbol{E}^q_{ij} y^q_{ij},$$

where y^q_{ij} denotes a free real variable and $\boldsymbol{E}^q_{ij}$ an $n^q \times n^q$ matrix with 1 at the (i,j)th and (j,i)th elements and 0 elsewhere. Then substitute it into the general SDP.

A general SDP:

min.	a linear function in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$ $(1 \le q \le t)$
sub.to	linear equalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,
	linear inequalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,
	linear (matrix) inequalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,
	$x_1, \ldots, x_k \in \mathbb{R}$ (free real variables),
	$\boldsymbol{X}^q \succeq \boldsymbol{O}$ $(1 \le q \le t)$ (psd matrix variables).

A general SDP:

min.	a linear function in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$ $(1 \le q \le t)$
sub.to	linear equalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,
	linear inequalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,
	linear (matrix) inequalities in $x_1, \ldots, x_k$ and $\boldsymbol{X}^q$,
	$x_1, \ldots, x_k \in \mathbb{R}$ (free real variables),
	$\boldsymbol{X}^q \succeq \boldsymbol{O}$ $(1 \le q \le t)$ (psd matrix variables).

'An LMI standard form with equality constraints':

min	a linear function in $y_1, \ldots, y_\ell$
sub.to	linear equalities in $y_1, \ldots, y_\ell$,
	linear (matrix) inequalities in $y_1, \ldots, y_\ell$,
	$y_1, \ldots, y_\ell \in \mathbb{R}$ (free real variables).

- Take the dual $\Rightarrow$ an eq. standard form with free variables.
- We can apply existing software; CSDP, PENON, SDPA, SDPT3 and SeDuMi to this primal-dual pair.

Exercise. Transform the SDP

$$\min \quad w + \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \bullet \boldsymbol{X} \quad \text{sub.to} \quad \begin{pmatrix} \boldsymbol{X} & \begin{matrix} 2 \\ 1 \end{matrix} \\ \begin{matrix} 2 & 1 \end{matrix} & w \end{pmatrix} \succeq \boldsymbol{O}.$$

to an LMI standard form SDP

$$\begin{array}{ll} \min & w + 2y_1 + 2y_2 + 3y_3 \\ \text{sub.to} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_3 \\ & + \begin{pmatrix} \boldsymbol{O} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} w + \begin{pmatrix} \boldsymbol{O} & \begin{matrix} 2 \\ 1 \end{matrix} \\ \begin{matrix} 2 & 1 \end{matrix} & 0 \end{pmatrix} \succeq \boldsymbol{O}. \end{array}$$

Contents

Eigenvalues of a symmetric matrix $\boldsymbol{A}$

$$\begin{aligned} \text{the max. eigenvalue} &= \min\{\lambda : \lambda \boldsymbol{I} \succeq \boldsymbol{A}\} \\ &= \min\{\lambda : \lambda \boldsymbol{I} - \boldsymbol{A} \succeq \boldsymbol{O}\}. \\ \text{the min. eigenvalue} &= \max\{\lambda : \boldsymbol{A} - \lambda \boldsymbol{I} \succeq \boldsymbol{O}\}. \end{aligned}$$

- We can formulate many engineering problems involving eigenvalues of symmetric matrices via SDPs.
- A Linear Matrix inequality (LMI) $\boldsymbol{A}(\cdot) \succeq \boldsymbol{O}$, where $\boldsymbol{A}(\cdot)$ is a linear mapping in matrix and/or vector variables can be formulated in

  maximize λ subject to $\boldsymbol{A}(\cdot) - \lambda \boldsymbol{I} \succeq \boldsymbol{O}$.

  For example,
  $$\boldsymbol{A}(\boldsymbol{X}) = \begin{pmatrix} \boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{X} + \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{X}\boldsymbol{B} + \boldsymbol{C}^T\boldsymbol{D} \\ \boldsymbol{B}^T\boldsymbol{X} + \boldsymbol{D}^T\boldsymbol{C} & \boldsymbol{D}^T\boldsymbol{D} - \boldsymbol{I} \end{pmatrix} \succeq \boldsymbol{O}.$$

For LMIs, see

[6] S. Boyd, L. El Ghaui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

The Schur complement. Let
$\boldsymbol{A} \in \mathbb{S}^k$, positive definite, $\boldsymbol{X} \in \mathbb{R}^{k\times\ell}$, $\boldsymbol{Y} \in \mathbb{S}^\ell$.
Then
$$\underbrace{\boldsymbol{Y} - \boldsymbol{X}^T\boldsymbol{A}^{-1}\boldsymbol{X} \succeq \boldsymbol{O}}_{\text{quadratic in } \boldsymbol{X}} \Leftrightarrow \underbrace{\begin{pmatrix} \boldsymbol{A} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}}_{\text{linear in } \boldsymbol{X}}.$$

Proof:
$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{Y} - \boldsymbol{X}^T\boldsymbol{A}^{-1}\boldsymbol{X} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{A}^{-1}\boldsymbol{X} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix}^T \begin{pmatrix} \boldsymbol{A} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{Y} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{A}^{-1}\boldsymbol{X} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix}$$

Hence
$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O} \Leftrightarrow \begin{pmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{Y} - \boldsymbol{X}^T\boldsymbol{A}^{-1}\boldsymbol{X} \end{pmatrix} \succeq \boldsymbol{O}.$$
$$\Updownarrow \ \boldsymbol{A} \text{ is positive definite.}$$
$$\boldsymbol{Y} - \boldsymbol{X}^T\boldsymbol{A}^{-1}\boldsymbol{X} \succeq \boldsymbol{O}.$$

The Schur complement. Let
$\boldsymbol{A} \in \mathbb{S}^k$, positive definite, $\boldsymbol{X} \in \mathbb{R}^{k\times\ell}$, $\boldsymbol{Y} \in \mathbb{S}^\ell$.
Then
$$\underbrace{\boldsymbol{Y} - \boldsymbol{X}^T\boldsymbol{A}^{-1}\boldsymbol{X} \succeq \boldsymbol{O}}_{\text{quadratic in } \boldsymbol{X}} \Leftrightarrow \underbrace{\begin{pmatrix} \boldsymbol{A} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}}_{\text{linear in } \boldsymbol{X}}.$$

The Schur complement. Let

$$A \in \mathbb{S}^k, \text{ positive definite}, \quad X \in \mathbb{R}^{k\times\ell},\ Y \in \mathbb{S}^\ell.$$

Then

$$\underbrace{Y - X^T A^{-1} X \succeq O}_{\text{quadratic in } X} \quad \Leftrightarrow \quad \underbrace{\begin{pmatrix} A & X \\ X^T & Y \end{pmatrix} \succeq O.}_{\text{linear in } X}$$

- When $A = I$, $Y - X^T X \succeq O \quad \Leftrightarrow \quad \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq O.$
- When $A = I$, $X = x \in \mathbb{R}^k$ and $Y = y \in \mathbb{R}$,
$$y - x^T x \geq 0 \quad \Leftrightarrow \quad \begin{pmatrix} I & x \\ x^T & y \end{pmatrix} \succeq O.$$
- When $A = Iy$, $X = x \in \mathbb{R}^k$ and $Y = y \in \mathbb{R}$,
$$y - \sqrt{x^T x} \geq 0 \Leftrightarrow y^2 - x^T x \geq 0,\ y \geq 0 \Leftrightarrow \begin{pmatrix} Iy & x \\ x^T & y \end{pmatrix} \succeq O.$$
$$(y - x^T x / y \geq 0 \text{ if } y > 0)$$

A quasi-convex optimization problem

$$\min \frac{(Lx - c)^T (Lx - c)}{d^T x} \text{ sub.to } Ax \geq b.$$

Here $L \in \mathbb{R}^{k\times n}$, $c \in \mathbb{R}^k$, $d \in \mathbb{R}^n$, $A \in \mathbb{R}^{\ell\times n}$, $b \in \mathbb{R}^\ell$, and $d^T x > 0$ for $\forall$ feasible $x \in \mathbb{R}^n$.

$$\Updownarrow$$

$$\min \zeta \text{ sub.to } \zeta \geq \frac{(Lx - c)^T (Lx - c)}{d^T x},\ Ax \geq b.$$

$$\Updownarrow \ \zeta - \frac{(Lx - c)^T (Lx - c)}{d^T x} \geq 0 \Leftrightarrow \begin{pmatrix} (d^T x) I & Lx - c \\ (Lx - c)^T & \zeta \end{pmatrix} \succeq O.$$

SDP: $\min \zeta$ sub.to $\begin{pmatrix} d^T x I & Lx - c \\ (Lx - c)^T & \zeta \end{pmatrix} \succeq O,\ Ax \geq b.$

$$\Downarrow$$

SOCP

Matrix approximation problem — 1

Let F_p be an $k \times \ell$ matrix $(0 \leq p \leq m)$. Approximate the matrix F_0 as a linear combination of F_p $(1 \leq p \leq m)$;

$$\text{minimize } \{\|F(x)\| : x \in \mathbb{R}^m\},$$

where $F(x) = F_0 - \sum_{p=1}^m F_p x_p$ for $\forall x = (x_1, \ldots, x_m)^T$.

- Which norm?

$$\|A\|_\infty = \max\{|A_{ij}| : 1 \leq i \leq k,\ 1 \leq j \leq \ell\} \text{ (the } \infty \text{ norm)}$$

$$\|A\|_F = \left(\sum_{i=1}^k \sum_{j=1}^\ell A_{ij}^2\right)^{1/2} \text{ (the Frobenius norm)}$$

$$\|A\|_{L_2} = \max_{\|u\|_2=1} \|Au\| = \left(\text{the maximum eigenvalue of } A^T A\right)^{1/2}$$

(the L_2 operator norm).

Matrix approximation problem — 2

Let F_p be an $k \times \ell$ matrix $(0 \leq p \leq m)$. Approximate the matrix F_0 as a linear combination of F_p $(1 \leq p \leq m)$;

$$\text{minimize } \{\|F(x)\| : x \in \mathbb{R}^m\},$$

where $F(x) = F_0 - \sum_{p=1}^m F_p x_p$ for $\forall x = (x_1, \ldots, x_m)^T$.

Matrix approximation problem — 2

Let $\boldsymbol{F}_p$ be an $k \times \ell$ matrix $(0 \le p \le m)$. Approximate the matrix $\boldsymbol{F}_0$ as a linear combination of $\boldsymbol{F}_p$ $(1 \le p \le m)$;

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\| : \boldsymbol{x} \in \mathbb{R}^m\},$$

where $\boldsymbol{F}(\boldsymbol{x}) \;=\; \boldsymbol{F}_0 - \sum_{p=1}^m \boldsymbol{F}_p x_p$ for $\forall \boldsymbol{x} = (x_1, \dots, x_m)^T$.

$$\|\boldsymbol{A}\|_\infty \;=\; \max\{|A_{ij}| : 1 \le i \le k,\ 1 \le j \le \ell\} \text{ (the } \infty \text{ norm)}$$

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\|_\infty : \boldsymbol{x} \in \mathbb{R}^m\}$$

$$\Downarrow$$

$$\text{minimize } \max\{|F_{ij}(\boldsymbol{x})| : 1 \le i \le k,\ 1 \le j \le \ell\}$$

$$\Downarrow$$

$$\text{minimize } \zeta \text{ sub.to } -\zeta \le F_{ij}(\boldsymbol{x}) \le \zeta \ (1 \le i \le k,\ 1 \le j \le \ell)$$

LP (Linear Programming)

Matrix approximation problem — 3

Let $\boldsymbol{F}_p$ be an $k \times \ell$ matrix $(0 \le p \le m)$. Approximate the matrix $\boldsymbol{F}_0$ as a linear combination of $\boldsymbol{F}_p$ $(1 \le p \le m)$;

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\| : \boldsymbol{x} \in \mathbb{R}^m\},$$

where $\boldsymbol{F}(\boldsymbol{x}) \;=\; \boldsymbol{F}_0 - \sum_{p=1}^m \boldsymbol{F}_p x_p$ for $\forall \boldsymbol{x} = (x_1, \dots, x_m)^T$.

Matrix approximation problem — 3

Let $\boldsymbol{F}_p$ be an $k \times \ell$ matrix $(0 \le p \le m)$. Approximate the matrix $\boldsymbol{F}_0$ as a linear combination of $\boldsymbol{F}_p$ $(1 \le p \le m)$;

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\| : \boldsymbol{x} \in \mathbb{R}^m\},$$

where $\boldsymbol{F}(\boldsymbol{x}) \;=\; \boldsymbol{F}_0 - \sum_{p=1}^m \boldsymbol{F}_p x_p$ for $\forall \boldsymbol{x} = (x_1, \dots, x_m)^T$.

$$\|\boldsymbol{A}\|_F \;=\; \left(\sum_{i=1}^k \sum_{j=1}^\ell A_{ij}^2\right)^{1/2} \quad \text{(the Frobenius norm)}$$

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\|_F : \boldsymbol{x} \in \mathbb{R}^m\}$$

$$\Downarrow$$

$$\text{minimize}_{\boldsymbol{x} \in \mathbb{R}^m} \ \|\boldsymbol{F}(\boldsymbol{x})\|_F^2 \equiv \sum_{i=1}^k \sum_{j=1}^\ell F_{ij}(\boldsymbol{x})^2$$

the least square problem

convex QP (quadratic Programming)

Matrix approximation problem — 4

Let $\boldsymbol{F}_p$ be an $k \times \ell$ matrix $(0 \le p \le m)$. Approximate the matrix $\boldsymbol{F}_0$ as a linear combination of $\boldsymbol{F}_p$ $(1 \le p \le m)$;

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\| : \boldsymbol{x} \in \mathbb{R}^m\},$$

where $\boldsymbol{F}(\boldsymbol{x}) \;=\; \boldsymbol{F}_0 - \sum_{p=1}^m \boldsymbol{F}_p x_p$ for $\forall \boldsymbol{x} = (x_1, \dots, x_m)^T$.

Matrix approximation problem — 4

Let $\boldsymbol{F}_p$ be an $k \times \ell$ matrix $(0 \le p \le m)$. Approximate the matrix $\boldsymbol{F}_0$ as a linear combination of $\boldsymbol{F}_p$ $(1 \le p \le m)$;

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\| : \boldsymbol{x} \in \mathbb{R}^m\},$$

where $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}_0 - \sum_{p=1}^m \boldsymbol{F}_p x_p$ for $\forall \boldsymbol{x} = (x_1, \ldots, x_m)^T$.

$$\|\boldsymbol{A}\|_{L_2} = \max_{\|\boldsymbol{u}\|_2 = 1} \|\boldsymbol{A}\boldsymbol{u}\| = (\text{the maximum eigenvalue of } \boldsymbol{A}^T\boldsymbol{A})^{1/2}$$

(the L_2 operator norm)

$$\text{minimize } \{\|\boldsymbol{F}(\boldsymbol{x})\|_{L_2} : \boldsymbol{x} \in \mathbb{R}^m\}$$

$$\Downarrow$$

minimize "the maximum eigenvalue of $\boldsymbol{F}(\boldsymbol{x})^T\boldsymbol{F}(\boldsymbol{x})$"

$$\Downarrow$$

minimize λ subject to $\lambda \boldsymbol{I} - \boldsymbol{F}(\boldsymbol{x})^T\boldsymbol{F}(\boldsymbol{x}) \succeq \boldsymbol{O}$

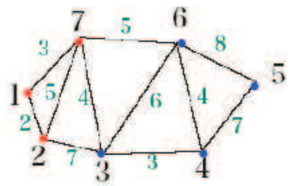
$\Downarrow$ the Schur complement

minimize λ subject to $\begin{pmatrix} \boldsymbol{I} & \boldsymbol{F}(\boldsymbol{x}) \\ \boldsymbol{F}(\boldsymbol{x})^T & \lambda \boldsymbol{I} \end{pmatrix} \succeq \boldsymbol{O}$ (SDP)

Max-cut problem — 1

The max-cut problem: Let $G = (N, E)$ be an undirected graph, and w_{ij} be weights of an edge $\{i, j\} \in E$.
For $\forall K \subset N$, let $\delta(K)$ denote $\{\{i,j\} : i \in K,\ j \notin K\}$ (the cut determined by K) and $w(\delta(K)) = \sum_{\{i,j\} \in \delta(K)} w_{ij}$.

Max-cut problem: max $w(\delta(K))$ s.t. $K \subset N$.



$N = \{1,2,3,4,5,6,7\}$, $w_{12} = w_{21} = 2, \ldots$

$*K = \{1,2,7\} \Rightarrow \delta(K) = \{\{2,3\},\{3,7\},\{6,7\}\}$

$w(\delta(K)) = 7+4+5 = 16$

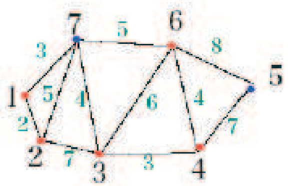
$K = \{1,2,3,4,6\} \Rightarrow \delta(K) = \{\{1,7\},\{2,7\},\{3,7\},\{4,5\},\{5,6\},\{6,7\}\}$

$w(\delta(K)) = 3+5+4+7+8+5 = 32$

Max-cut problem — 2

The max-cut problem: Let $G = (N, E)$ be an undirected graph, and w_{ij} be weights of an edge $\{i, j\} \in E$.
For $\forall K \subset N$, let $\delta(K)$ denote $\{\{i,j\} : i \in K,\ j \notin K\}$ (the cut determined by K) and $w(\delta(K)) = \sum_{\{i,j\} \in \delta(K)} w_{ij}$.

Max-cut problem: max $w(\delta(K))$ s.t. $K \subset N$.



$N = \{1,2,3,4,5,6,7\}$, $w_{12} = w_{21} = 2, \ldots$

$K = \{1,2,7\} \Rightarrow \delta(K) = \{\{2,3\},\{3,7\},\{6,7\}\}$

$w(\delta(K)) = 7+4+5 = 16$

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$w(\delta(K)) = 3+5+4+7+8+5 = 32$

Max-cut problem — 3

The max-cut problem: Let $G = (N, E)$ be an undirected graph, and w_{ij} be weights of an edge $\{i, j\} \in E$.
For $\forall K \subset N$, let $\delta(K)$ denote $\{\{i,j\} : i \in K,\ j \notin K\}$ (the cut determined by K) and $w(\delta(K)) = \sum_{\{i,j\} \in \delta(K)} w_{ij}$.

Max-cut problem: max $w(\delta(K))$ s.t. $K \subset N$.

Let $w_{ij} = 0$ if $\{i,j\} \notin E$, and let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$; $x_i = \begin{cases} 1 & \text{if } i \in K, \\ -1 & \text{otherwise.} \end{cases}$ Then $w(\delta(K)) = \dfrac{1}{2}\displaystyle\sum_{i<j} w_{ij}(1 - x_i x_j) = \dfrac{1}{4}\sum_{i=1}^n \sum_{j=1}^n w_{ij}(1 - x_i x_j) = \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}$, where $c_{ij} = -w_{ij}/4$ $(i \ne j)$ and $c_{ii} = \sum_{j=1}^n w_{ij}$.

Exercise. Verify the identity $\dfrac{1}{4}\displaystyle\sum_{i=1}^n \sum_{j=1}^n w_{ij}(1 - x_i x_j) = \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}$.

$$\Downarrow$$

Max-cut problem — 4

Max-cut prob. $\Leftrightarrow$ $c^* = \max \boldsymbol{C} \bullet \boldsymbol{x}^T\boldsymbol{x}$ s.t. $x_i^2 = 1 \ (i \in N)$

$\Longrightarrow$ relaxation

SDP: $\hat{c} = \max \boldsymbol{C} \bullet \boldsymbol{X}$ s.t. $X_{ii} = 1 \ (i \in N), \ \boldsymbol{X} \succeq \boldsymbol{O}$

- $c^* \le \hat{c}$ Exercise 18. Show this inequality.
- How do we construct a cut from an opt.sol. $\widehat{\boldsymbol{X}}$ of SDP?

Max-cut problem — 4

Max-cut prob. $\Leftrightarrow$ $c^* = \max \boldsymbol{C} \bullet \boldsymbol{x}^T\boldsymbol{x}$ s.t. $x_i^2 = 1 \ (i \in N)$

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SDP: $\hat{c} = \max \boldsymbol{C} \bullet \boldsymbol{X}$ s.t. $X_{ii} = 1 \ (i \in N), \ \boldsymbol{X} \succeq \boldsymbol{O}$

- $c^* \le \hat{c}$ Exercise 18. Show this inequality.
- How do we construct a cut from an opt.sol. $\widehat{\boldsymbol{X}}$ of SDP?

Step 1. Factorize $\widehat{\boldsymbol{X}}$ s.t. $\widehat{\boldsymbol{X}} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)^T (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$.

Step 2. Choose a vector ξ randomly from the unit sphere $\{\eta \in \mathbb{R}^n : \|\eta\| = 1\}$; hence ξ is a random variable vector.

Step 3. Let

$$\boldsymbol{x}_i(\xi) = \begin{cases} 1 & \text{if } \boldsymbol{v}_i^T\xi > 0, \\ -1 & \text{otherwise} \end{cases} \quad \text{or} \quad K(\xi) = \{i \in N : \boldsymbol{v}_i^T\xi > 0\}$$

$$\Downarrow$$

$$\frac{E(w(\delta(K(\xi))))}{\text{the value } c^* \text{ of max-cut}} \ge 0.878$$

Contents

A primal-dual pair of LPs

(P)	min	$\boldsymbol{a}_0 \cdot \boldsymbol{x}$	s.t.	$\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p), \ \boldsymbol{x} \ge \mathbf{0}$.
(D)	max	$\sum_{p=1}^m b_p y_p$	s.t.	$\sum_{p=1}^m \boldsymbol{a}_p y_p + s = \boldsymbol{a}_0, \ \mathbb{R}^n \ni s \ge \mathbf{0}$.

Weak duality

$$\text{LP} \ : \ \boldsymbol{x} \cdot s = \boldsymbol{a}_0 \cdot \boldsymbol{x} - \sum_{j=1}^m b_p y_p \ge 0 \text{ for } \forall \text{ feasible } \boldsymbol{x}, \ \boldsymbol{y}, \ s.$$

$$\text{SDP} \ : \ \boldsymbol{X} \bullet S = \boldsymbol{A}_0 \bullet \boldsymbol{X} - \sum_{j=1}^m b_p y_p \ge 0 \text{ for } \forall \text{ feasible } \boldsymbol{X}, \ \boldsymbol{y}, \ \boldsymbol{S}.$$

Exercise. Prove the weak duality

$$\boldsymbol{X} \bullet S = \boldsymbol{A}_0 \bullet \boldsymbol{X} - \sum_{j=1}^m b_p y_p \ge 0 \text{ for } \forall \text{ feasible } \boldsymbol{X}, \ \boldsymbol{y}, \ \boldsymbol{S}.$$

A primal-dual pair of SDPs

(P)	min.	$\boldsymbol{A}_0 \bullet \boldsymbol{X}$	sub.to	$\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \succeq \boldsymbol{O}$.
(D)	max.	$\sum_{p=1}^m b_p y_p$	sub.to	$\sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \boldsymbol{S} \succeq \boldsymbol{O}$.

A primal-dual pair of LPs

(P)	min	$\boldsymbol{a}_0 \cdot \boldsymbol{x}$	s.t.	$\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p), \ \boldsymbol{x} \geq \mathbf{0}$.
(D)	max	$\sum_{p=1}^m b_p y_p$	s.t.	$\sum_{p=1}^m \boldsymbol{a}_p y_p + s = \boldsymbol{a}_0, \ \mathbb{R}^n \ni s \geq \mathbf{0}$.

Strong duality: If $\exists$ feasible $(\boldsymbol{x}, \ \boldsymbol{y}, \ s)$ $(\boldsymbol{x} \geq 0, \ \boldsymbol{y} \geq \mathbf{0})$ then

$$\text{LP} \ : \ \bar{\boldsymbol{x}} \cdot \bar{s} = \boldsymbol{a}_0 \cdot \boldsymbol{x} - \sum_{j=1}^m b_p \bar{y}_p = 0 \text{ at } \forall \text{ optimal } (\bar{\boldsymbol{x}}, \ \bar{\boldsymbol{y}}, \ \bar{s}).$$

If $\exists$ interior feasible $(\boldsymbol{X}, \ \boldsymbol{y}, \ \boldsymbol{S})$ $(\boldsymbol{X} \succ \boldsymbol{O}, \ \boldsymbol{S} \succ \boldsymbol{O})$ then

$$\text{SDP} \ : \ \bar{\boldsymbol{X}} \bullet \bar{\boldsymbol{S}} = \boldsymbol{A}_0 \bullet \bar{\boldsymbol{X}} - \sum_{j=1}^m b_p \bar{y}_p = 0 \text{ at } \forall \text{ optimal } (\bar{\boldsymbol{X}}, \ \bar{\boldsymbol{y}}, \ \bar{\boldsymbol{S}}).$$

- For the strong duality, "$\exists$ int. feasible $(\boldsymbol{X}, \boldsymbol{y}, \boldsymbol{S})$ $(\boldsymbol{X} \succ \boldsymbol{O}, \boldsymbol{S} \succ \boldsymbol{O})$" is necessary! $\Rightarrow$ an example, next

A primal-dual pair of SDPs

(P)	min.	$\boldsymbol{A}_0 \bullet \boldsymbol{X}$	sub.to	$\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \succeq \boldsymbol{O}$.
(D)	max.	$\sum_{p=1}^m b_p y_p$	sub.to	$\sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \boldsymbol{S} \succeq \boldsymbol{O}$.

Example [45]: "$\exists$ interior feasible $(\boldsymbol{X}, \boldsymbol{y}, \boldsymbol{S})$ $(\boldsymbol{X} \succ \boldsymbol{O}, \boldsymbol{S} \succ \boldsymbol{O})$" is necessary!

$$\text{(P) min } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \boldsymbol{X}$$

$$\text{sub.to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet \boldsymbol{X} = 0, \ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \bullet \boldsymbol{X} = 2, \boldsymbol{X} \succeq \boldsymbol{O}.$$

or

$$\text{(P) min } \ X_{33} \ \text{ sub.to } \ X_{11} = 0, X_{12} + X_{21} + 2X_{33} = 2, \boldsymbol{X} \succeq \boldsymbol{O}.$$

Exercise 6. Show that the objective value $X_{33} = 1$ if $\boldsymbol{X}$ is feasible.

$$\text{(D) max } \ 2y_2$$

$$\text{sub.to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} y_2 \preceq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

or

$$\text{(D) min } \ 2y_2 \ \text{ sub.to } \begin{pmatrix} -y_1 & -y_2 & 0 \\ -y_2 & 0 & 0 \\ 0 & 0 & 1 - 2y_2 \end{pmatrix} \succeq \boldsymbol{O}.$$

Exercise. Show that the objective value $2y_2 = 0$ if (y_1, y_2) is feasible.

A primal-dual pair of SDPs

(P)	min.	$\boldsymbol{A}_0 \bullet \boldsymbol{X}$	sub.to	$\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \succeq \boldsymbol{O}$.
(D)	max.	$\sum_{p=1}^m b_p y_p$	sub.to	$\sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \boldsymbol{S} \succeq \boldsymbol{O}$.

The KKT optimality condition

$\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m), \ \sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0,$
$\mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}, \ \mathbb{S}^n \ni \boldsymbol{S} \succeq \boldsymbol{O}, \ \boldsymbol{X}\boldsymbol{S} = \boldsymbol{O}$ (complementarity).

$\boldsymbol{O} = \boldsymbol{X}\boldsymbol{S} = \boldsymbol{S}\boldsymbol{X} \Rightarrow \boldsymbol{X}$ and $\boldsymbol{S}$ are commutative; hence
$\Downarrow$ $\exists$ orthogonal $\boldsymbol{P} \in \mathbb{R}^{n \times n}$; $\boldsymbol{P}^T \boldsymbol{X} \boldsymbol{P} = \text{diag}\ (\lambda_1, \ldots, \lambda_n)$,
$\boldsymbol{P}^T \boldsymbol{S} \boldsymbol{P} = \text{diag}\ (\nu_1, \ldots, \nu_n)$

$\boldsymbol{O} = \boldsymbol{X}\boldsymbol{S} = P^T \boldsymbol{X} P P^T \boldsymbol{S} P = \text{diag}\ (\lambda_1, \ldots, \lambda_n) \text{diag}\ (\nu_1, \ldots, \nu_n)$,
$P^T (\boldsymbol{X} + \boldsymbol{S}) P = \text{diag}\ (\lambda_1, \ldots, \lambda_n) + \text{diag}\ (\nu_1, \ldots, \nu_n)$.

$\Downarrow$

$\lambda_i \geq 0, \ \nu_i \geq 0, \ \lambda_i \nu_i = 0 \ (1 \leq i \leq n)$ (complementarity),
$X + S \succ O \Leftrightarrow \lambda_i + \nu_i > 0 \ (1 \leq i \leq n)$ (strict comp.).

LP: $x_i \geq 0, s_i \geq 0, x_i s_i = 0 \ (\forall i)$ (comp.), $x_i + s_i > 0 \ (\forall i)$

An equality standard form

(P) min. $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (1 \leq p \leq m), \ \boldsymbol{X} \succeq \boldsymbol{O}.$

An equality standard form with free variables

(P) min. $\boldsymbol{A}_0 \bullet \boldsymbol{X} + \boldsymbol{d}_0^T \boldsymbol{z}$

sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} + \boldsymbol{d}_p^T \boldsymbol{z} = b_p \ (1 \leq p \leq m),$

$\mathbb{S}^n \ni \boldsymbol{X} \succeq \boldsymbol{O}, \ \boldsymbol{z} \in \mathbb{R}^\ell$ (a free vector variable).

Here $\boldsymbol{d}_p \in \mathbb{R}^\ell \ (0 \leq p \leq m).$

$\Updownarrow$ dual

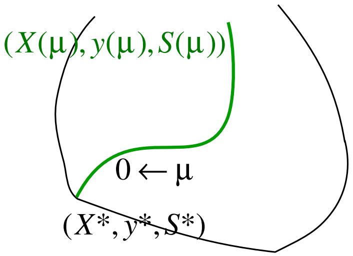
An LMI standard form with equality constraints

(D) max. $\displaystyle\sum_{p=1}^m b_p y_p$

sub.to $\displaystyle\sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \mathbb{S}^n \ni \boldsymbol{S} \succeq \boldsymbol{O}, \ \sum_{p=1}^m \boldsymbol{d}_p y_p = \boldsymbol{d}_0.$

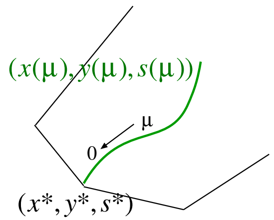
Contents

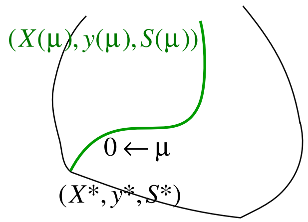
- There exists a trajectory with the parameter $\mu > 0$ in the primal-dual space which leads to a primal-dual pair of optimal solutions of SDP as $\mu \to 0$. We call this trajectory the central trajectory.
- The primal-dual interior-point method numerically traces the central trajectory.



LP:	P	min	$\boldsymbol{a}_0 \cdot \boldsymbol{x}$	s.t.	$\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p = 1), \ \boldsymbol{x} \in \mathbb{R}^n_+$
	D	max	$\sum_{p=1}^m b_p y_p$	s.t.	$\sum_{p=1}^m \boldsymbol{a}_p y_p + \boldsymbol{s} = \boldsymbol{a}_0, \ s \in \mathbb{R}^n_+$
SDP:	P	min	$\boldsymbol{A}_0 \bullet \boldsymbol{X}$	s.t.	$\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \in \mathcal{S}^n_+$
	D	max	$\sum_{p=1}^m b_p y_p$	s.t.	$\sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \boldsymbol{S} \in \mathcal{S}^n_+$

- Basic idea of the primal-dual interior-point method:
  Trace the central trajectory $\to$ an opt. sol. in the p-d space.





- How do we define the central trajectory?
- How do we numerically trace the central trajectory?

LP: P min $\boldsymbol{a}_0 \cdot \boldsymbol{x}$ s.t. $\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p = 1), \ \boldsymbol{x} \in \mathbb{R}^n_+$
D max $\sum_{p=1}^m b_p y_p$ s.t. $\sum_{p=1}^m \boldsymbol{a}_p y_p + \boldsymbol{s} = \boldsymbol{a}_0, \ s \in \mathbb{R}^n_+$

SDP: P min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ s.t. $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \in \mathcal{S}^n_+$
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LP: P min $\boldsymbol{a}_0 \cdot \boldsymbol{x}$ s.t. $\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p = 1), \ \boldsymbol{x} \in \mathbb{R}^n_+$
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D max $\sum_{p=1}^m b_p y_p$ s.t. $\sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \boldsymbol{S} \in \mathcal{S}^n_+$

- A log barrier to be away from the boundary $-\sum_{i=1}^m \log x_i$.

$\boldsymbol{x} \in$ the boundary of $\mathbb{R}^n_+$ $\Leftrightarrow x_i = 0 \ (i = 1, \ldots, n)$.
$\boldsymbol{x} \in$ the interior of $\mathbb{R}^n_+ \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\} \Leftrightarrow x_i > 0 \ (i = 1, \ldots, n)$.

- A log barrier to be away from the boundary $-\log\det \boldsymbol{X}$.

$\boldsymbol{X} \in$ the interior of $\mathcal{S}^n_+ \equiv \{\boldsymbol{X} \in \mathbb{S}^n : \boldsymbol{X} \succeq \boldsymbol{O}\} \Leftrightarrow$ det $\boldsymbol{X} > 0$.
$\boldsymbol{X} \in$ the boundary of $\mathcal{S}^n_+$ $\Leftrightarrow$ det $\boldsymbol{X}$ = 0.

LP: P min $\boldsymbol{a}_0 \cdot \boldsymbol{x}$ s.t. $\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p = 1), \ \boldsymbol{x} \in \mathbb{R}^n_+$
D max $\sum_{p=1}^m b_p y_p$ s.t. $\sum_{p=1}^m \boldsymbol{a}_p y_p + \boldsymbol{s} = \boldsymbol{a}_0, \ s \in \mathbb{R}^n_+$

SDP: P min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ s.t. $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \in \mathcal{S}^n_+$
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A primal-dual pair of LPs with logarithmic barrier terms, $\mu > 0$

P(μ) min $\boldsymbol{a}_0 \cdot \boldsymbol{x} - \mu \sum_{i=1}^m \log x_i$ s.t. $\boldsymbol{a}_p \cdot \boldsymbol{x} = b_p \ (\forall p), \ \boldsymbol{x} > \boldsymbol{0}$

D(μ) max $\sum_{p=1}^m b_p y_p + \mu \sum_{i=1}^m \log s_i$ s.t. $\sum_{p=1}^m \boldsymbol{a}_p y_p + \boldsymbol{s} = \boldsymbol{a}_0, \ \boldsymbol{s} > \boldsymbol{0}$

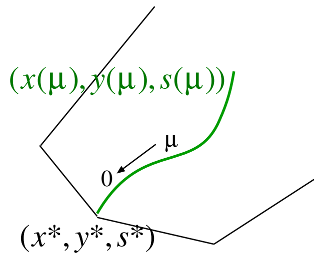
A primal-dual pair of SDPs with logarithmic barrier terms, $\mu > 0$

P(μ) min $\boldsymbol{A}_0 \bullet \boldsymbol{X} - \mu \log\det \boldsymbol{X}$ s.t. $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \succ \boldsymbol{O}$
D(μ) max $\sum_{p=1}^m b_p y_p + \mu \log\det \boldsymbol{S}$
s.t. $\sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \boldsymbol{S} \succ \boldsymbol{O}$

A primal-dual pair of LPs with logarithmic barrier terms, $\mu > 0$

$$\begin{array}{lll} \text{P}(\mu) & \min & \boldsymbol{a}_0 \cdot \boldsymbol{x} - \mu \sum_{i=1}^{m} \log x_i \ \text{s.t.}\ \boldsymbol{a}_p \cdot \boldsymbol{x} = b_p\ (\forall p),\ \boldsymbol{x} > \boldsymbol{0} \\ \text{D}(\mu) & \max & \sum_{p=1}^{m} b_p y_p + \mu \sum_{i=1}^{m} \log s_i \ \text{s.t.}\ \sum_{p=1}^{m} \boldsymbol{a}_p y_p + \boldsymbol{s} = \boldsymbol{a}_0,\ \boldsymbol{s} > \boldsymbol{0} \end{array}$$

- For every $\mu > 0$, (P(μ),D(μ)) has a unique opt.sol. $(\boldsymbol{x}(\mu), \boldsymbol{y}(\mu), \boldsymbol{s}(\mu))$, which converges an opt. sol. of (P,D).



- $C = \{(\boldsymbol{x}(\mu), \boldsymbol{y}(\mu), \boldsymbol{s}(\mu)) : \mu > 0\}$: the central trajectory.

A primal-dual pair of SDPs with logarithmic barrier terms, $\mu > 0$

$$\begin{array}{ll} \text{P}(\mu) & \min\ \boldsymbol{A}_0 \bullet \boldsymbol{X} - \mu \log\det \boldsymbol{X}\ \text{s.t.}\ \boldsymbol{A}_p \bullet \boldsymbol{X} = b_p\ (\forall p),\ \boldsymbol{X} \succ \boldsymbol{O} \\ \text{D}(\mu) & \max\ \sum_{p=1}^{m} b_p y_p + \mu \log\det \boldsymbol{S} \\ & \text{s.t.}\ \sum_{p=1}^{m} \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0,\ \boldsymbol{S} \succ \boldsymbol{O} \end{array}$$

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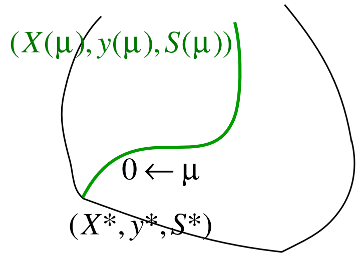

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A primal-dual pair of SDPs with logarithmic barrier terms, $\mu > 0$

$$\begin{array}{ll} \text{P}(\mu) & \min\ \boldsymbol{A}_0 \bullet \boldsymbol{X} - \mu \log\det \boldsymbol{X}\ \text{s.t.}\ \boldsymbol{A}_p \bullet \boldsymbol{X} = b_p\ (\forall p),\ \boldsymbol{X} \succ \boldsymbol{O} \\ \text{D}(\mu) & \max\ \sum_{p=1}^{m} b_p y_p + \mu \log\det \boldsymbol{S} \\ & \text{s.t.}\ \sum_{p=1}^{m} \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0,\ \boldsymbol{S} \succ \boldsymbol{O} \end{array}$$

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A primal-dual pair of SDPs with logarithmic barrier terms, $\mu > 0$

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- For every $\mu > 0$, (P(μ),D(μ)) has a unique opt.sol. $(\boldsymbol{X}(\mu), \boldsymbol{y}(\mu), \boldsymbol{S}(\mu))$, which converges an opt. sol. of (P,D).
- For $\forall \mu > 0$, the obj. function of P(μ) is convex in $\boldsymbol{X}$.
- For $\forall \mu > 0$, the obj. function of D(μ) is concave in $(\boldsymbol{y}, \boldsymbol{S})$.
- For every $\mu > 0$, $(\boldsymbol{X}(\mu), \boldsymbol{y}(\mu), \boldsymbol{S}(\mu))$ is characterized as the Karush-Kuhn-Tucker optimality condition

$$\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p\ (\forall p), \sum_{p=1}^{m} \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0,$$
$$\boldsymbol{X} \succ 0,\ \boldsymbol{S} \succ 0,\ \boldsymbol{X}\boldsymbol{S} = \mu \boldsymbol{I}.$$

- A modified Newton method to trace the central trajectory $C = \{(\boldsymbol{X}(\mu), \boldsymbol{y}(\mu), \boldsymbol{S}(\mu)) : \mu > 0\}$.

Contents

Some existing numerical methods for SDPs

- IPMs (Interior-point methods)
  - Primal-dual scaling, CSDP(Borchers[7]), SDPA(Fujisawa-K-Nakata-Yamashita[49]), SDPT3(Toh-Todd-Tutuncu[42]), SeDuMi(Sturm[37])
  - Dual scaling, DSDP(Benson-Ye-Zhang[3])
- Nonlinear programming approaches
  - Spectral bundle method(Helmberg-Rendl[17])
  - Gradient-based log-barrier method(Burer-Monteiro[9])
  - PENON(Kocvara [19]) — Augmented Lagrangian
  - Saddle point mirror-prox algorithm (Lu-Nemirovski-Monteiro[26])

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- Medium scale (e.g. $n, \ m \leq 5000$) and high accuracy.
- Large scale (e.g., $n, m \geq 10,000$) and low accuracy.

• Parallel implementation:
SDPA ⇒ SDPARA(Y-F-K[49]), SDPARA-C(N-Y-F-K[31])
DSDP ⇒ PDSDP(Benson[2]), CSDP ⇒ Borchers-Young[8]
Spectral bundle method ⇒ Nayakkankuppam[32]

Optimization Technology Center
http://www.ece.northwestern.edu/OTC/
⇓
NEOS Solvers
http://www-neos.mcs.anl.gov/neos/solvers/index.html
⇓
• Semidefinite Programming

software	lang.	method
csdp	c	p-d ipm
pensdp	matalb	augmented Lagrangian
sdpa	c++	p-d ipm
sdpt3	matlab	p-d ipm
sedumi	matlab	p-d ipm , self-dual embedding
. . .	. . .	. . .

- Binary and/or source codes are available.
- SDPA sparse format for all packages, matlab interface.
- Online solver — submit your SDP problem through Internet.

Some remarks on software packages.

- SDPs are more difficult to solve than LPs.
  - Degeneracy, no interior points in primal or dual SDPs.
  - Large scale problems.
- More accuracy requires more cpu time.
- Some package can solve SDPs faster with low accuracy.
- Sparse structure of SDPs.
- Some SDPs can be solved faster and/or more accurately by one package, but other SDPs by some other else.

Try some software packages that fit your problem.

SDPA Online Solver
http://sdpara.r.dendai.ac.jp/portal/

- SDPA on a single cpu.
- SDPARA on Opteron 32 cpu (1.4GHz), Xeon 16cpu (2.8GHz).
- SDPARA-C on Opteron 32 cpu (1.4GHz), Xeon 16cpu (2.8GHz).
- Submit your problem and choose one of the packages.

Contents

$\mathcal{P}$: min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \in \mathcal{S}^n_+$
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From quantum chemistry, Fukuda et al. [13], Zhao et al. [51].

problem	m	n	#blocks	the sizes of largest blocks
O	7230	5990	22	[1450, 1450, 450, ...]
HF	15018	10146	22	[2520, 2520, 792, ...]
CH_3N	20709	12802	22	[3211, 3211, 1014, ...]

Parallel computation: cpu time in second

# of processors	16	64	128	256
O	14250.6	4453.3	3281.1	2951.6
HF	*	*	26797.1	20780.7
CH_3N	*	*	57034.8	45488.9

$\mathcal{P}$: min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \in \mathcal{S}^n_+$
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Large-size SDPs by SDPARA-C [31] (64 CPUs)

3 types of test Problems:

(a) SDP relaxations of max. cut problems on lattice graphs with size 10×1000, 10×2000 and 10×4000.

(b) SDP relaxations of max. clique problems on lattice graphs with size 10×500, 10×1000 and 10×2000.

(c) Norm minimization problems

$$\text{min.} \ \left\| \boldsymbol{F}_0 - \sum_{i=1}^{10} \boldsymbol{F}_i y_i \right\| \ \text{sub.to} \ y_i \in \mathbb{R} \ (i = 1, 2, \ldots, 10)$$

where $\boldsymbol{F}_i : 10 \times 9990, \ 10 \times 19990$ or 10×39990 and $\|\boldsymbol{G}\|$ = the square root of the max. eigenvalue of $\boldsymbol{G}^T\boldsymbol{G}$.

In all cases, the aggregate sparsity pattern consists of one block and is very sparse.

$\mathcal{P}$: min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \in \mathcal{S}^n_+$
$\mathcal{D}$: max $\sum_{p=1}^m b_p y_p$ sub.to $\sum_{p=1}^m \boldsymbol{A}_p y_p + \boldsymbol{S} = \boldsymbol{A}_0, \ \boldsymbol{S} \in \mathcal{S}^n_+$

Large-size SDPs by SDPARA-C (64 CPUs)

$\mathcal{P}$: min $\boldsymbol{A}_0 \bullet \boldsymbol{X}$ sub.to $\boldsymbol{A}_p \bullet \boldsymbol{X} = b_p \ (\forall p), \ \boldsymbol{X} \in \mathcal{S}^n_+$
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Large-size SDPs by SDPARA-C (64 CPUs)

	Problem	n	m	time (s)	memory (MB)
	Cut(10×1000)	10000	10000	274.3	126
(a)	Cut(10×2000)	20000	20000	1328.2	276
	Cut(10×4000)	40000	40000	7462.0	720
	Clique(10×500)	5000	9491	639.5	119
(b)	Clique(10×1000)	10000	18991	3033.2	259
	Clique(10×2000)	20000	37991	15329.0	669
	Norm(10×9990)	10000	11	409.5	164
(c)	Norm(10×19990)	20000	11	1800.9	304
	Norm(10×39990)	40000	11	7706.0	583

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