

Enclosing Ellipsoids and Elliptic Cylinders
of Semialgebraic Sets and Their Application
to Error Bounds in Polynomial Optimization

Masakazu Kojima and Makoto Yamashita
Tokyo Institute of Technology

November 2010

At National Cheng-Kung University, Tainan

.. - p.1/37

Outline

- 1 Problem and Some Formulations
- 2 Theory: Lifting and SDP Relaxation
- 3 Numerical Results
- 4 Applications to the Sensor Network Localization Problem
- 5 Concluding Remarks

.. - p.2/37

Outline

- 1 **Problem and Some Formulations**
- 2 Theory: Lifting and SDP Relaxation
- 3 Numerical Results
- 4 Applications to the Sensor Network Localization Problem
- 5 Concluding Remarks

.. - p.3/37

Problem

Given a nonempty compact semialgebraic subset

$$F = \{\mathbf{x} \in \mathbb{R}^n : f_k(\mathbf{x}) \geq 0 \ (k = 1, 2, \dots, m)\}$$

of \mathbb{R}^n , find a “small” ellipsoid enclosing F . Here $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$
denotes a polynomial ($k = 1, 2, \dots, m$).

- “small” needs to be specified.

.. - p.4/37

Formulation 1: Minimum volume ellipsoid

F : a nonempty compact semialgebraic subset of \mathbb{R}^n .
 $\mathcal{E}(M, c) \equiv \{x \in \mathbb{R}^n : (x - c)^T M (x - c) \leq 1\}$.
 minimize volume of $\mathcal{E}(M, c)$
 sub.to $F \subset \mathcal{E}(M, c), M \succ O, c \in \mathbb{R}^n$.

- The most popular in theory
- F consists of a finite number of points \Rightarrow lots of studies \supset (Khachiyan's method 1996)
- Ideal but too difficult in general

Formulation 2: Nie and Demmel 2005

F : a nonempty compact semialgebraic subset of \mathbb{R}^n .
 $\mathcal{E}(P^{-1}, c) \equiv \{x \in \mathbb{R}^n : (x - c)^T P^{-1} (x - c) \leq 1\}$.
 minimize Trace P
 sub.to $F \subset \mathcal{E}(P^{-1}, c), P \succ O, c \in \mathbb{R}^n$.
 \Leftarrow SOS (Sum Of Squares) relaxation

- A little more general to include parameters.
- Theoretical convergence.
- Still very expensive to apply it to large-scale problems.
 - The SOS relaxation problem becomes a dense problem.

\Rightarrow Less expensive formulation: Fix the shape of the ellipsoid and minimize the size — Ours, next

Our Formulation:

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, shape). Define
 $\varphi(x, c) \equiv (x - c)^T M (x - c), \forall x, \forall c \in \mathbb{R}^n$ (center),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (size).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n

A min. enclosing ellipsoidal set : $\gamma^* = \min_{\gamma, c} \{\gamma : F \subset E(c, \gamma)\}$.

Our Formulation:

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, shape). Define
 $\varphi(x, c) \equiv (x - c)^T M (x - c), \forall x, \forall c \in \mathbb{R}^n$ (center),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (size).

Our Formulation:

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**). Define

$$\varphi(\mathbf{x}, \mathbf{c}) \equiv (\mathbf{x} - \mathbf{c})^T \mathbf{M} (\mathbf{x} - \mathbf{c}), \forall \mathbf{x}, \forall \mathbf{c} \in \mathbb{R}^n \text{ (center),}$$

Ellipsoidal set $E(\mathbf{c}, \gamma) \equiv \{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}, \mathbf{c}) \leq \gamma\}$, $\forall \gamma > 0$ (**size**).

Application to error bounds in Polynomial Optimization

POP : $f_0^* = \min f_0(\mathbf{x})$ subject to $f_k(\mathbf{x}) \geq 0$ ($k = 1, 2, \dots, p$).
Here $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$: a polynomial ($k = 0, 1, \dots, p$).

Suppose that $\hat{f}_0 \geq f_0^*$ or $\hat{f}_0 = f_0(\hat{\mathbf{x}})$ for \exists feasible $\hat{\mathbf{x}}$. Let

$$F = \{\mathbf{x} \in \mathbb{R}^n : f_k(\mathbf{x}) \geq 0, (k = 1, 2, \dots, p), f_0(\mathbf{x}) \leq \hat{f}_0\}$$

$F \subset E(\mathbf{c}, \gamma) \implies E(\mathbf{c}, \gamma)$ contains all opt. solutions of POP.

$$\begin{aligned} \mathbf{M} = \mathbf{I} &\implies \|\mathbf{x} - \mathbf{c}\| \leq \sqrt{\gamma} \text{ for } \forall \text{ opt. sol. } \mathbf{x} \\ \mathbf{M} = \text{diag}(1, 0, \dots, 0) &\implies |x_1 - c_1| \leq \sqrt{\gamma} \text{ for } \forall \text{ opt. sol. } \mathbf{x} \end{aligned}$$

.. p.8/37

Our Formulation:

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**). Define

$$\varphi(\mathbf{x}, \mathbf{c}) \equiv (\mathbf{x} - \mathbf{c})^T \mathbf{M} (\mathbf{x} - \mathbf{c}), \forall \mathbf{x}, \forall \mathbf{c} \in \mathbb{R}^n \text{ (center),}$$

Ellipsoidal set $E(\mathbf{c}, \gamma) \equiv \{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}, \mathbf{c}) \leq \gamma\}$, $\forall \gamma > 0$ (**size**).

Application to error bounds in Polynomial Optimization

POP : $f_0^* = \min f_0(\mathbf{x})$ subject to $f_k(\mathbf{x}) \geq 0$ ($k = 1, 2, \dots, p$).
Here $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$: a polynomial ($k = 0, 1, \dots, p$).

Suppose that $\hat{f}_0 \geq f_0^*$ or $\hat{f}_0 = f_0(\hat{\mathbf{x}})$ for \exists feasible $\hat{\mathbf{x}}$. Let

$$F = \{\mathbf{x} \in \mathbb{R}^n : f_k(\mathbf{x}) \geq 0, (k = 1, 2, \dots, p), f_0(\mathbf{x}) \leq \hat{f}_0\}$$

$F \subset E(\mathbf{c}, \gamma) \implies E(\mathbf{c}, \gamma)$ contains all opt. solutions of POP.

$$\begin{aligned} \mathbf{M} = \mathbf{I} &\implies \|\mathbf{x} - \mathbf{c}\| \leq \sqrt{\gamma} \text{ for } \forall \text{ opt. sol. } \mathbf{x} \\ \mathbf{M} = \text{diag}(1, 0, \dots, 0) &\implies |x_1 - c_1| \leq \sqrt{\gamma} \text{ for } \forall \text{ opt. sol. } \mathbf{x} \end{aligned}$$

- This method can be combined with the SDP relaxation (Lasserre '01) and its sparse variant (Waki et al. '06).

.. p.8/37

Outline

- 1 Problem and Some Formulations
- 2 Theory: Lifting and SDP Relaxation
- 3 Numerical Results
- 4 Applications to the Sensor Network Localization Problem
- 5 Concluding Remarks

.. p.9/37

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),

$$\varphi(\mathbf{x}, \mathbf{c}) \equiv (\mathbf{x} - \mathbf{c})^T \mathbf{M} (\mathbf{x} - \mathbf{c}), \forall \mathbf{x}, \forall \mathbf{c} \in \mathbb{R}^n \text{ (center),}$$

Ellipsoidal set $E(\mathbf{c}, \gamma) \equiv \{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}, \mathbf{c}) \leq \gamma\}$, $\forall \gamma > 0$ (**size**).

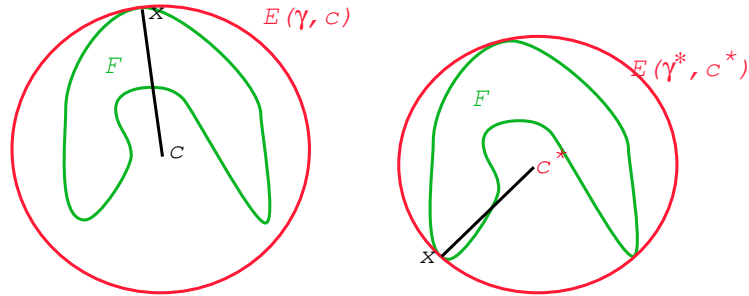
F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

.. p.10/37

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(\mathbf{x}, \mathbf{c}) \equiv (\mathbf{x} - \mathbf{c})^T M (\mathbf{x} - \mathbf{c}), \forall \mathbf{x}, \forall \mathbf{c} \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(\mathbf{c}, \gamma) \equiv \{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}, \mathbf{c}) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

min-max formulation $\gamma^* = \min_{\mathbf{c} \in \mathbb{R}^n} \max_{\mathbf{x} \in F} \varphi(\mathbf{x}, \mathbf{c}) = \max_{\mathbf{x} \in F} \varphi(\mathbf{x}, \mathbf{c}^*)$.

Suppose that M = the 2×2 identity matrix



$\bullet \varphi(\mathbf{x}, \mathbf{c}) = M \bullet \mathbf{x}\mathbf{x}^T - 2\mathbf{x}^T M \mathbf{c} + \mathbf{c}^T M \mathbf{c}, \forall \mathbf{x}, \forall \mathbf{c} \in \mathbb{R}^n$.

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(\mathbf{x}, \mathbf{c}) = M \bullet \mathbf{x}\mathbf{x}^T - 2\mathbf{x}^T M \mathbf{c} + \mathbf{c}^T M \mathbf{c}, \forall \mathbf{x}, \forall \mathbf{c} \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(\mathbf{c}, \gamma) \equiv \{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}, \mathbf{c}) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

min-max formulation $\gamma^* = \min_{\mathbf{c} \in \mathbb{R}^n} \max_{\mathbf{x} \in F} \varphi(\mathbf{x}, \mathbf{c}) = \max_{\mathbf{x} \in F} \varphi(\mathbf{x}, \mathbf{c}^*)$.

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(\mathbf{x}, \mathbf{c}) = M \bullet \mathbf{x}\mathbf{x}^T - 2\mathbf{x}^T M \mathbf{c} + \mathbf{c}^T M \mathbf{c}, \forall \mathbf{x}, \forall \mathbf{c} \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(\mathbf{c}, \gamma) \equiv \{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}, \mathbf{c}) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

min-max formulation $\gamma^* = \min_{\mathbf{c} \in \mathbb{R}^n} \max_{\mathbf{x} \in F} \varphi(\mathbf{x}, \mathbf{c}) = \max_{\mathbf{x} \in F} \varphi(\mathbf{x}, \mathbf{c}^*)$.

Lifting \Rightarrow

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(\mathbf{x}, \mathbf{c}) = M \bullet \mathbf{x}\mathbf{x}^T - 2\mathbf{x}^T M \mathbf{c} + \mathbf{c}^T M \mathbf{c}, \forall \mathbf{x}, \forall \mathbf{c} \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(\mathbf{c}, \gamma) \equiv \{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}, \mathbf{c}) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

min-max formulation $\gamma^* = \min_{\mathbf{c} \in \mathbb{R}^n} \max_{\mathbf{x} \in F} \varphi(\mathbf{x}, \mathbf{c}) = \max_{\mathbf{x} \in F} \varphi(\mathbf{x}, \mathbf{c}^*)$.

Lifting \Rightarrow

Define $\psi(\mathbf{x}, \mathbf{W}, \mathbf{c}) \equiv M \bullet \mathbf{W} - 2\mathbf{x}^T M \mathbf{c} + \mathbf{c}^T M \mathbf{c}$,
 $C^* \equiv$ the convex hull of $\{(\mathbf{x}, \mathbf{x}\mathbf{x}^T) \in \mathbb{R}^n \times \mathbb{S}^n : \mathbf{x} \in F\}$.

convex-linear min-max formulation $\gamma^* = \min_{\mathbf{c} \in \mathbb{R}^n} \left(\max_{(\mathbf{x}, \mathbf{W}) \in C^*} \psi(\mathbf{x}, \mathbf{W}, \mathbf{c}) \right)$.

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

min-max formulation $\gamma^* = \min_{c \in \mathbb{R}^n} \max_{x \in F} \varphi(x, c) = \max_{x \in F} \varphi(x, c^*)$. Lifting \Rightarrow

Define $\psi(x, W, c) \equiv M \bullet W - 2x^T M c + c^T M c$,
 $C^* \equiv$ the convex hull of $\{(x, xx^T) \in \mathbb{R}^n \times \mathbb{S}^n : x \in F\}$.

convex-linear min-max formulation $\gamma^* = \min_{c \in \mathbb{R}^n} \left(\max_{(x, W) \in C^*} \psi(x, W, c) \right)$.

linear-convex max-min problem $\gamma^* = \max_{(x, W) \in C^*} \min_{c \in \mathbb{R}^n} \psi(x, W, c)$.

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

min-max formulation $\gamma^* = \min_{c \in \mathbb{R}^n} \max_{x \in F} \varphi(x, c) = \max_{x \in F} \varphi(x, c^*)$. Lifting \Rightarrow

Define $\psi(x, W, c) \equiv M \bullet W - 2x^T M c + c^T M c$,
 $C^* \equiv$ the convex hull of $\{(x, xx^T) \in \mathbb{R}^n \times \mathbb{S}^n : x \in F\}$.

convex-linear min-max formulation $\gamma^* = \min_{c \in \mathbb{R}^n} \left(\max_{(x, W) \in C^*} \psi(x, W, c) \right)$.

linear-convex max-min problem $\gamma^* = \max_{(x, W) \in C^*} \min_{c \in \mathbb{R}^n} \psi(x, W, c)$.

$\min_{c \in \mathbb{R}^n} M \bullet W - 2x^T M c + c^T M c \quad \Downarrow \quad c^* = x : \text{a minimizer}$

concave maximization $\gamma^* = \max_{(x, W) \in C^*} M \bullet W - x^T M x$.

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

concave maximization $\gamma^* = \max_{(x, W) \in C^*} M \bullet W - x^T M x$.

Here $C^* \equiv$ the convex hull of $\{(x, xx^T) \in \mathbb{R}^n \times \mathbb{S}^n : x \in F\}$.

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

concave maximization $\gamma^* = \max_{(x, W) \in C^*} M \bullet W - x^T M x$.

Here $C^* \equiv$ the convex hull of $\{(x, xx^T) \in \mathbb{R}^n \times \mathbb{S}^n : x \in F\}$.

- Relax the intractable C^* by a tractable convex \hat{C} ;

$\Downarrow L \equiv \left\{ (x, W) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & x^T \\ x & W \end{pmatrix} \succeq O \right\} \supset \hat{C} \supset C^*.$

- Describe \hat{C} in terms of LMIs.

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

concave maximization $\gamma^* = \max_{(x, W) \in C^*} M \bullet W - x^T M x$.

Here $C^* \equiv$ the convex hull of $\{(x, xx^T) \in \mathbb{R}^n \times \mathbb{S}^n : x \in F\}$.

- Relax the intractable C^* by a tractable convex \hat{C} ;

$\Downarrow L \equiv \left\{ (x, W) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & x^T \\ x & W \end{pmatrix} \succeq O \right\} \supset \hat{C} \supset C^*.$

- Describe \hat{C} in terms of LMIs.

SDP-SOCP $\hat{\gamma} = \max_{(x, W) \in \hat{C}} M \bullet W - x^T M x \Rightarrow \gamma^* \leq \hat{\gamma}.$

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

concave maximization $\gamma^* = \max_{(x, W) \in C^*} M \bullet W - x^T M x$.

Here $C^* \equiv$ the convex hull of $\{(x, xx^T) \in \mathbb{R}^n \times \mathbb{S}^n : x \in F\}$.

- Relax the intractable C^* by a tractable convex \hat{C} ;

$\Downarrow L \equiv \left\{ (x, W) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & x^T \\ x & W \end{pmatrix} \succeq O \right\} \supset \hat{C} \supset C^*.$

- Describe \hat{C} in terms of LMIs.

SDP-SOCP $\hat{\gamma} = \max_{(x, W) \in \hat{C}} M \bullet W - x^T M x \Rightarrow \gamma^* \leq \hat{\gamma}.$

- Under an assumption, $\{C^k : \text{described in terms of LMIs}\};$
 $L \supset C^k \supset C^{k+1} \supset C^*$ and $\bigcap_k C^k = C^*$
 by using Lasserre's hierarchy of LMI relaxation '01.

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

concave maximization $\gamma^* = \max_{(x, W) \in C^*} M \bullet W - x^T M x$.

Here $C^* \equiv$ the convex hull of $\{(x, xx^T) \in \mathbb{R}^n \times \mathbb{S}^n : x \in F\}$.

- Relax the intractable C^* by a tractable convex \hat{C} ;

$\Downarrow L \equiv \left\{ (x, W) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & x^T \\ x & W \end{pmatrix} \succeq O \right\} \supset \hat{C} \supset C^*.$

- Describe \hat{C} in terms of LMIs.

SDP-SOCP $\hat{\gamma} = \max_{(x, W) \in \hat{C}} M \bullet W - x^T M x \Rightarrow \gamma^* \leq \hat{\gamma}.$

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).
 F : a nonempty compact semialgebraic subset of \mathbb{R}^n .

concave maximization $\gamma^* = \max_{(x, W) \in C^*} M \bullet W - x^T M x$.

Here $C^* \equiv$ the convex hull of $\{(x, xx^T) \in \mathbb{R}^n \times \mathbb{S}^n : x \in F\}$.

- Relax the intractable C^* by a tractable convex \hat{C} ;

$\Downarrow L \equiv \left\{ (x, W) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & x^T \\ x & W \end{pmatrix} \succeq O \right\} \supset \hat{C} \supset C^*$

- Describe \hat{C} in terms of LMIs.

SDP-SOCP $\hat{\gamma} = \max_{(x, W) \in \hat{C}} M \bullet W - x^T M x \Rightarrow \gamma^* \leq \hat{\gamma}$.

- When \hat{C} is described in terms of **sparse** LMIs, take M which fits **their sparsity**.
 \Rightarrow a **sparse SDP-SOCP** which we can solve efficiently.

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).

QOP case

$F = \{x \in \mathbb{R}^n : \alpha_k + 2b_k^T x + x^T Q_k x \geq 0 (1 \leq k \leq p)\}$
 $= \left\{ x \in \mathbb{R}^n : \begin{pmatrix} \alpha_k & b_k^T \\ b_k & Q_k \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \geq 0 (1 \leq k \leq p) \right\},$

$M \in \mathbb{S}_+^n$ ($n \times n$ positive semidefinite matrices, **shape**),
 $\varphi(x, c) = M \bullet xx^T - 2x^T M c + c^T M c, \forall x, \forall c \in \mathbb{R}^n$ (**center**),
 Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \forall \gamma > 0$ (**size**).

QOP case

$F = \{x \in \mathbb{R}^n : \alpha_k + 2b_k^T x + x^T Q_k x \geq 0 (1 \leq k \leq p)\}$
 $= \left\{ x \in \mathbb{R}^n : \begin{pmatrix} \alpha_k & b_k^T \\ b_k & Q_k \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \geq 0 (1 \leq k \leq p) \right\},$

Let

$\hat{C} = \left\{ (x, W) : \begin{pmatrix} \alpha_k & b_k^T \\ b_k & Q_k \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & W \end{pmatrix} \geq 0 (1 \leq k \leq p), \right. \\ \left. \begin{pmatrix} 1 & x^T \\ x & W \end{pmatrix} \succeq O \right\},$

SDP-SOCP $\hat{\gamma} = \max_{(x, W) \in \hat{C}} M \bullet W - x^T M x = M \bullet \hat{W} - \hat{c}^T M \hat{c}$

$\implies F \subset E(\hat{c}, \hat{\gamma})$.

Outline

- 1 Problem and Some Formulations
- 2 Theory: Lifting and SDP Relaxation
- 3 Numerical Results**
- 4 Applications to the Sensor Network Localization Problem
- 5 Concluding Remarks

.. p.15/37

Outline

- 1 Problem and Some Formulations
 - 2 Theory: Lifting and SDP Relaxation
 - 3 Numerical Results**
 - 4 Applications to the Sensor Network Localization Problem
 - 5 Concluding Remarks
- SparsePOP (Waki et al. '08) for constructing sparse SDP relaxation problems of POPs.
 - SeDuMi1.21 (Sturm, Polik '09) for solving SDP relaxation problems to compute **an approx. opt. sol. of POPs** and for solving SDP-SOCs to compute error bounds.
 - MATLAB Optimization Toolbox to refine **the approx. opt. sol.** obtained by SeDuMi for constrained optimization problems.
 - 2.8GHz Intel Xeon with 4GB Memory.

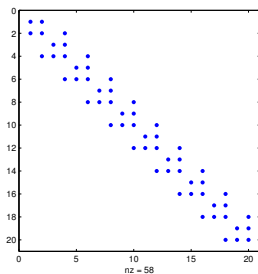
.. p.15/37

Unconstrained min. of ChainedWood function $f_C(\mathbf{x})$

$$f_C(\mathbf{x}) = 1 + \sum_{i \in J} (100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2 + (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2)$$

Here $J = \{1, 3, 5, \dots, n - 3\}$ and n is a multiple of 4.

Sparsity pattern of the Hessian matrix



$n = 20$, 20×20 matrix
no. of nonzeros = $\frac{58}{400}$

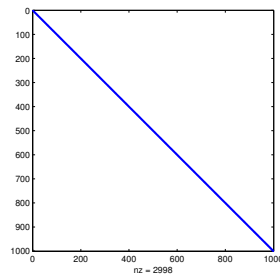
- Sparse enough to solve larger scale problems.

.. p.16/37

Unconstrained min. of ChainedWood function $f_C(\mathbf{x})$

$$f_C(\mathbf{x}) = 1 + \sum_{i \in J} (100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2 + (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2)$$

Here $J = \{1, 3, 5, \dots, n - 3\}$ and n is a multiple of 4.



$n = 1000$, 1000×1000 matrix
no. of nonzeros = $\frac{2,988}{1,000,000}$

.. p.17/37

Unconstrained min. of ChainedWood function $f_C(\mathbf{x})$

$$f_C(\mathbf{x}) = 1 + \sum (100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2 + (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2)$$

Here $J = \{1, 3, 5, \dots, n - 3\}$ and n is a multiple of 4.

$M =$ the $n \times n$ identity matrix.

n	RelObjErr at $\hat{\mathbf{x}}$	E.Time for $\hat{\mathbf{x}}$	Error bound E.time $\sqrt{\hat{\gamma}}/\ \hat{\mathbf{c}}\ $
1000	4.4e-4	2.4	4.7 4.9e-3
2000	8.8e-4	5.7	11.6 4.9e-3
4000	1.8e-3	14.6	30.3 1.5e-3

$\hat{\mathbf{x}} =$ an approx. optimal solution,

$$\text{RelObjErr} = \frac{|\text{lbd. for opt. val.} - f_C(\hat{\mathbf{x}})|}{|f_C(\hat{\mathbf{x}})|}$$

$$\|\mathbf{x} - \hat{\mathbf{c}}\|/\|\hat{\mathbf{c}}\| \leq \sqrt{\hat{\gamma}}/\|\hat{\mathbf{c}}\|, \forall \text{ global minimizer } \mathbf{x}$$

.. p.17/37

alkyl.gms from globallib

$$\begin{aligned} \min & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\ \text{sub.to} & x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0, \\ & x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574, \\ & 0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, -x_2x_9 + 10x_3 + x_6 = 0, \\ & -0.820x_2 + x_5 - 0.820x_6 = 0, x_1x_{11} - 3x_8 = -1.33, \\ & x_{10}x_{14} + 22.2x_{11} = 35.82, \text{lbd}_i \leq x_i \leq \text{ubd}_i \quad (i = 1, 2, \dots, 14). \end{aligned}$$

$$\frac{|\text{lbd for opt.val.} - \text{approx. opt.val } f_0(\hat{\mathbf{x}})|}{|\text{approx. opt.val } f_0(\hat{\mathbf{x}})|} = 6.7e-6$$

max error in equalities at $\hat{\mathbf{x}} = 5.2e-9$

$$F = \{\mathbf{x} \in \mathbb{R}^{14} : \text{feasible and } f_0(\mathbf{x}) \leq f_0(\hat{\mathbf{x}})\} \subset E(\hat{\mathbf{c}}, \hat{\gamma})$$

.. p.18/37

alkyl.gms from globallib

$$\begin{aligned} \min & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\ \text{sub.to} & x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0, \\ & x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574, \\ & 0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, -x_2x_9 + 10x_3 + x_6 = 0, \\ & -0.820x_2 + x_5 - 0.820x_6 = 0, x_1x_{11} - 3x_8 = -1.33, \\ & x_{10}x_{14} + 22.2x_{11} = 35.82, \text{lbd}_i \leq x_i \leq \text{ubd}_i \quad (i = 1, 2, \dots, 14). \end{aligned}$$

$$\frac{|\text{lbd for opt.val.} - \text{approx. opt.val } f_0(\hat{\mathbf{x}})|}{|\text{approx. opt.val } f_0(\hat{\mathbf{x}})|} = 6.7e-6$$

max error in equalities at $\hat{\mathbf{x}} = 5.2e-9$

$$F = \{\mathbf{x} \in \mathbb{R}^{14} : \text{feasible and } f_0(\mathbf{x}) \leq f_0(\hat{\mathbf{x}})\} \subset E(\hat{\mathbf{c}}, \hat{\gamma})$$

$$\mathbf{M} = \mathbf{I} \in \mathbb{S}^{14} \Rightarrow \hat{\mathbf{c}} = (1.7037030, 1.5847109, \dots), \sqrt{\hat{\gamma}} = 1.6e-4.$$

$$\|\mathbf{x} - \hat{\mathbf{c}}\| \leq \sqrt{\hat{\gamma}} \text{ for } \forall \text{ opt. sol. } \mathbf{x} \in \mathbb{R}^{14}.$$

.. p.18/37

alkyl.gms from globallib

$$\begin{aligned} \min & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\ \text{sub.to} & x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0, \\ & x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574, \\ & 0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, -x_2x_9 + 10x_3 + x_6 = 0, \\ & -0.820x_2 + x_5 - 0.820x_6 = 0, x_1x_{11} - 3x_8 = -1.33, \\ & x_{10}x_{14} + 22.2x_{11} = 35.82, \text{lbd}_i \leq x_i \leq \text{ubd}_i \quad (i = 1, 2, \dots, 14). \end{aligned}$$

$$\frac{|\text{lbd for opt.val.} - \text{approx. opt.val } f_0(\hat{\mathbf{x}})|}{|\text{approx. opt.val } f_0(\hat{\mathbf{x}})|} = 6.7e-6$$

max error in equalities at $\hat{\mathbf{x}} = 5.2e-9$

$$F = \{\mathbf{x} \in \mathbb{R}^{14} : \text{feasible and } f_0(\mathbf{x}) \leq f_0(\hat{\mathbf{x}})\} \subset E(\hat{\mathbf{c}}, \hat{\gamma})$$

$$\mathbf{M} = \mathbf{I} \in \mathbb{S}^{14} \Rightarrow \hat{\mathbf{c}} = (1.7037030, 1.5847109, \dots), \sqrt{\hat{\gamma}} = 1.6e-4.$$

$$\|\mathbf{x} - \hat{\mathbf{c}}\| \leq \sqrt{\hat{\gamma}} \text{ for } \forall \text{ opt. sol. } \mathbf{x} \in \mathbb{R}^{14}.$$

$$\mathbf{M} = \text{diag}(1, 0, \dots, 0) \in \mathbb{S}^{14} \Rightarrow \hat{\mathbf{c}}_1 = 1.7037017, \sqrt{\hat{\gamma}} = 1.0e-5.$$

$$|x_1 - \hat{\mathbf{c}}_1| \leq \sqrt{\hat{\gamma}} \text{ for } \forall \text{ opt. sol. } \mathbf{x} \in \mathbb{R}^{14}.$$

.. p.18/37

Nonconvex QPs from globalib

$M =$ the $n \times n$ identity matrix

Problem	n	RelObjErr	Error bound		E.time sdpa
			$\sqrt{\hat{\gamma}}$	$\sqrt{\hat{\gamma}}/\ \hat{c}\ $	
ex2_1_3	13	1.1e-9	4.9e-4	4.9e-4	0.5
ex2_1_5	10	3.5e-10	4.7e-4	1.7e-4	0.8
ex2_1_8	24	3.5e-9	5.4e-2	1.3e-3	9.5
ex9_1_2 [†]	10	1.8e-2	4.2	0.53	0.2
ex9_1_5 [†]	13	6.2e-2	4.7	1.0	0.3
ex9_2_3	16	2.8e-7	1.4e-2	2.6e-4	0.2

$$\text{RelObjErr} = \frac{|\text{approx. otp. val.} - \text{l. bd. for otp. val.}|}{|\text{approx. otp. val.}|}$$

$$\|x - \hat{c}\| \leq \sqrt{\hat{\gamma}}, \forall \text{ global minimizer } x$$

[†] : multiple solutions

...p.19/37

More details on ex9_1_2[†]

$$\begin{aligned} \min. \quad & -x_1 - 3x_2 \\ \text{sub. to} \quad & 5 \text{ linear equations in } x_j \ (j = 1, 2, \dots, 10), \\ & x_3x_7 = 0, \ x_4x_8 = 0, \ x_5x_9 = 0, \ x_6x_{10} = 0, \\ & 0 \leq x_j \leq 5 \ (j = 1, 2, \dots, 10). \end{aligned}$$

$$\begin{aligned} M &= \text{diag}(\text{the } i\text{th unit coordinate vector}) \ (i = 3, 4, 5, 6, 8, 9) \\ &\Rightarrow |x_i - \hat{c}_i| \leq \sqrt{\hat{\gamma}} \ \text{for } \forall \text{ opt. sol. } x \end{aligned}$$

i	\hat{c}_i	$\sqrt{\hat{\gamma}}$	at \forall opt. sol.
3	2.9995	0.0089	$\Rightarrow x_3 > 0, x_7 = 0$
4	0.0002	0.0279	
5	0.0009	0.0148	
6	4.0002	0.0123	$\Rightarrow x_6 > 0, x_{10} = 0$
8	1.000	1.0001	
9	3.000	2.0004	$\Rightarrow x_9 > 0, x_5 = 0$

Fixing $x_5 = x_7 = x_{10} = 0$, we obtain the reduced problem \Rightarrow

...p.20/37

Reduced ex9_1_2[†] with fixing $x_5 = x_7 = x_{10} = 0$

$$\begin{aligned} \min. \quad & -x_1 - 3x_2 \\ \text{sub. to} \quad & 5 \text{ linear equations in } x_j \ (j = 1, 2, 3, 4, 6, 8, 9), \\ & x_4x_8 = 0, \ 0 \leq x_j \leq 5 \ (j = 1, 2, 3, 4, 6, 8, 9). \end{aligned}$$

$$\begin{aligned} M &= \text{diag}(\text{the } i\text{th unit coordinate vector}) \ (i = 1, 2, 3, 4, 6, 8, 9) \\ &\Rightarrow |x_i - \hat{c}_i| \leq \sqrt{\hat{\gamma}} \ \text{for } \forall \text{ opt. sol. } x \end{aligned}$$

i	\hat{c}_i	$\sqrt{\hat{\gamma}}$	i	\hat{c}_i	$\sqrt{\rho^*}$
1	4.0000	0.0002	2	4.0000	0.0002
3	3.0000	0.0006	4	0.0000	0.0006
6	4.0000	0.0004			
8	1.0000	1.0000	$\Rightarrow 0.0000 \leq x_8 \leq 2.0000$		
9	3.0000	2.0000	$\Rightarrow 1.0000 \leq x_9 \leq 5.0000$		

We can verify that the optimal solutions are:

$$x_1 = x_2 = x_6 = 4, \ x_3 = 3, \ x_4 = 0,$$

$$0 \leq x_8 = (x_9 - 1)/2 \leq 2, \ 1 \leq x_9 \leq 5.$$

...p.21/37

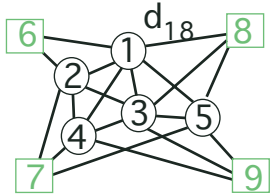
Outline

- 1 Problem and Some Formulations
- 2 Theory: Lifting and SDP Relaxation
- 3 Numerical Results
- 4 Applications to the Sensor Network Localization Problem
- 5 Concluding Remarks

...p.22/37

A Sensor Network Localization Problem with Exact Distance

$m = 5, n = 9.$
 $1, \dots, 5$: sensors
 $6, 7, 8, 9$: anchors



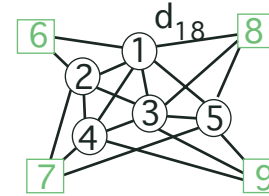
- Sensors' locations are unknown.
 - Anchors' locations are known.
 - A distance is given for \forall edge.
- Compute locations of sensors.
 \Rightarrow **Nonconvex QOPs**

... p.23/37

A Sensor Network Localization Problem with Exact Distance

$\mathbf{x}^p \in \mathbb{R}^2$: unknown location of sensors ($p = 1, 2, \dots, m$),
 $\mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2$: known location of anchors ($r = m + 1, \dots, n$),
 $d_{pq}^2 = \|\mathbf{x}^p - \mathbf{x}^q\|^2$ — given for $(p, q) \in \mathcal{E}$ (1)
 $\mathcal{E} = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\}$

$m = 5, n = 9.$
 $1, \dots, 5$: sensors
 $6, 7, 8, 9$: anchors



- Sensors' locations are unknown.
 - Anchors' locations are known.
 - A distance is given for \forall edge.
- Compute locations of sensors.
 \Rightarrow **Nonconvex QOPs**

... p.24/37

A Sensor Network Localization Problem with Exact Distance

$\mathbf{x}^p \in \mathbb{R}^2$: unknown location of sensors ($p = 1, 2, \dots, m$),
 $\mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2$: known location of anchors ($r = m + 1, \dots, n$),
 $d_{pq}^2 = \|\mathbf{x}^p - \mathbf{x}^q\|^2$ — given for $(p, q) \in \mathcal{E}$ (1)
 $\mathcal{E} = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\}$

... p.25/37

A Sensor Network Localization Problem with Exact Distance

$\mathbf{x}^p \in \mathbb{R}^2$: unknown location of sensors ($p = 1, 2, \dots, m$),
 $\mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2$: known location of anchors ($r = m + 1, \dots, n$),
 $d_{pq}^2 = \|\mathbf{x}^p - \mathbf{x}^q\|^2$ — given for $(p, q) \in \mathcal{E}$ (1)
 $\mathcal{E} = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\}$

- **FSDP by Biswas-Ye '06**, SDP relaxation of (1)
 - **Powerful in theory**;
 - FSDP** computes exact locations \mathbf{x}^p ($p = 1, 2, \dots, m$) if “(1) is uniquely localizable”
 - = “the rigidity of $G(\{1, 2, \dots, m\}, \mathcal{E})$ + a certain condition”.
 - But **expensive in computation**.
- **SFSDP by Kim, Kojima, Waki '09** = a sparse version of **FSDP** which is **as effective as FSDP in theory** but is **more efficient in computation**.

... p.25/37

A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned} \mathbf{x}^p \in \mathbb{R}^2 & : \text{ unknown location of sensors } (p = 1, 2, \dots, m), \\ \mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2 & : \text{ known location of anchors } (r = m + 1, \dots, n), \\ d_{pq}^2 & = \|\mathbf{x}^p - \mathbf{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \quad (1) \\ \mathcal{E} & = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

— p.26/37

A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned} \mathbf{x}^p \in \mathbb{R}^2 & : \text{ unknown location of sensors } (p = 1, 2, \dots, m), \\ \mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2 & : \text{ known location of anchors } (r = m + 1, \dots, n), \\ d_{pq}^2 & = \|\mathbf{x}^p - \mathbf{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \quad (1) \\ \mathcal{E} & = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

Numerical Results: 20,000 sensors randomly distributed in $[0, 1] \times [0, 1]$, 4 anchors at the corner and $\rho = 0.1$

σ	RMSD	SDPA E.time
0.0	6.9e-6	182.9
0.1	7.6e-3	403.0
0.2	1.1e-2	402.6

$$\begin{aligned} \text{RMSD} & = \left(\frac{1}{m} \sum_{p=1}^m \|\mathbf{x}^p - \mathbf{a}^p\| \right)^{1/2} \\ \mathbf{a}^p & : \text{ true location of } p \end{aligned}$$

$\sigma > 0 \Rightarrow d_{pq} = (1 + \xi) \times \text{true distance}$, different formulation:
 $\min \sum_{(p,q) \in \mathcal{E}} \|\|\mathbf{x}^p - \mathbf{x}^q\|^2 - d_{pq}^2\| \leftarrow \text{sparse SDP relaxation.}$
 Here ξ is chosen from $N(0, \sigma)$.

— p.26/37

A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned} \mathbf{x}^p \in \mathbb{R}^2 & : \text{ unknown location of sensors } (p = 1, 2, \dots, m), \\ \mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2 & : \text{ known location of anchors } (r = m + 1, \dots, n), \\ d_{pq}^2 & = \|\mathbf{x}^p - \mathbf{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \quad (1) \\ \mathcal{E} & = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

— p.27/37

A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned} \mathbf{x}^p \in \mathbb{R}^2 & : \text{ unknown location of sensors } (p = 1, 2, \dots, m), \\ \mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2 & : \text{ known location of anchors } (r = m + 1, \dots, n), \\ d_{pq}^2 & = \|\mathbf{x}^p - \mathbf{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \quad (1) \\ \mathcal{E} & = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

- Some numerical results of SFSD combined with our method for an ellipoidal set enclosing $F = \{(\mathbf{x}^1, \dots, \mathbf{x}^m) : d_{pq}^2 = \|\mathbf{x}^p - \mathbf{x}^q\|^2 \text{ for } (p, q) \in \mathcal{E}\}$.

— p.27/37

A Sensor Network Localization Problem with Exact Distance

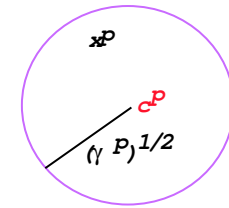
$$\begin{aligned} \mathbf{x}^p \in \mathbb{R}^2 & : \text{ unknown location of sensors } (p = 1, 2, \dots, m), \\ \mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2 & : \text{ known location of anchors } (r = m + 1, \dots, n), \\ d_{pq}^2 & = \|\mathbf{x}^p - \mathbf{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \quad (1) \\ \mathcal{E} & = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

— p.28/37

A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned} \mathbf{x}^p \in \mathbb{R}^2 & : \text{ unknown location of sensors } (p = 1, 2, \dots, m), \\ \mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2 & : \text{ known location of anchors } (r = m + 1, \dots, n), \\ d_{pq}^2 & = \|\mathbf{x}^p - \mathbf{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \quad (1) \\ \mathcal{E} & = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

Problem: For each sensor $p = 1, 2, \dots, m$, compute $\mathbf{c}^p \in \mathbb{R}^2$ and $\gamma^p > 0$ such that the distance from \mathbf{c}^p to its unknown location is bounded by $(\gamma^p)^{1/2}$.



— p.28/37

A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned} \mathbf{x}^p \in \mathbb{R}^2 & : \text{ unknown location of sensors } (p = 1, 2, \dots, m), \\ \mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2 & : \text{ known location of anchors } (r = m + 1, \dots, n), \\ d_{pq}^2 & = \|\mathbf{x}^p - \mathbf{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \quad (1) \\ \mathcal{E} & = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

— p.29/37

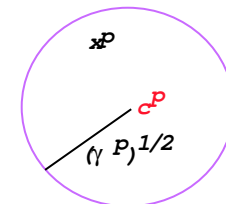
A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned} \mathbf{x}^p \in \mathbb{R}^2 & : \text{ unknown location of sensors } (p = 1, 2, \dots, m), \\ \mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^2 & : \text{ known location of anchors } (r = m + 1, \dots, n), \\ d_{pq}^2 & = \|\mathbf{x}^p - \mathbf{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \quad (1) \\ \mathcal{E} & = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

- When ρ is not large enough or \mathcal{E} does not contain enough number of edges, (1) is underdetermined and/or its SDP relaxation is too weak to locate all sensors uniquely.
- Our method + SFSDP computes $\mathbf{c}^p \in \mathbb{R}^2$ and $\gamma^p > 0$ for each sensor p such that the distance from \mathbf{c}^p to its unknown location \mathbf{x}^p is bounded by $(\gamma^p)^{1/2}$.

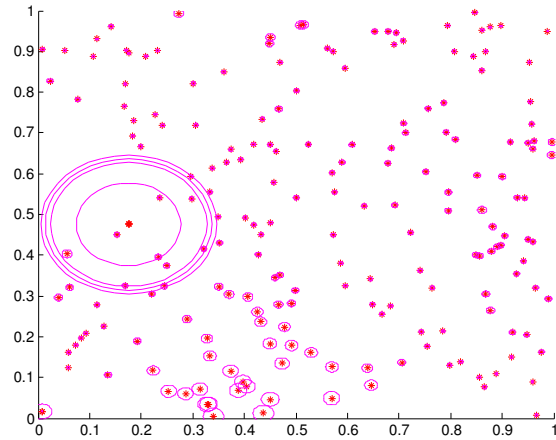
↓

- If $\gamma^p = 0$ then \mathbf{c}^p is the exact location of p (Biswas-Ye '06).



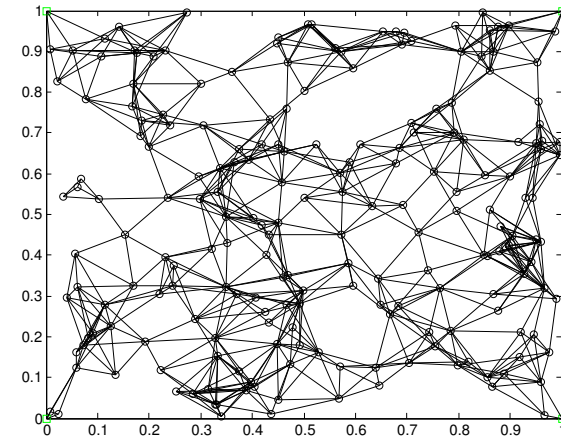
— p.29/37

$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$
anchors at the corner of $[0, 1]^2$, $\rho = 0.14$.



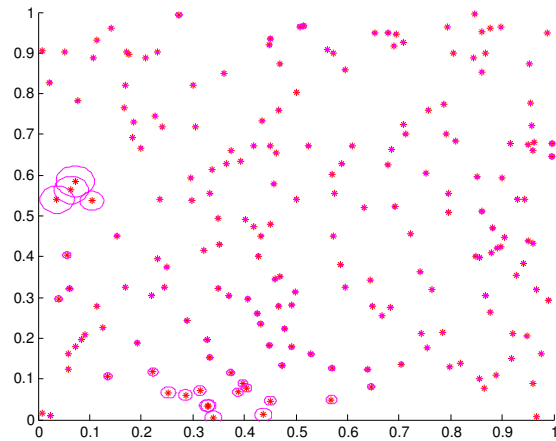
* : c^p = a computed location of sensor p .
the true location x^p of sensor p is within $(\gamma^p)^{1/2} \leq 0.18$ from c^p

$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$
anchors at the corner of $[0, 1]^2$, $\rho = 0.14$.



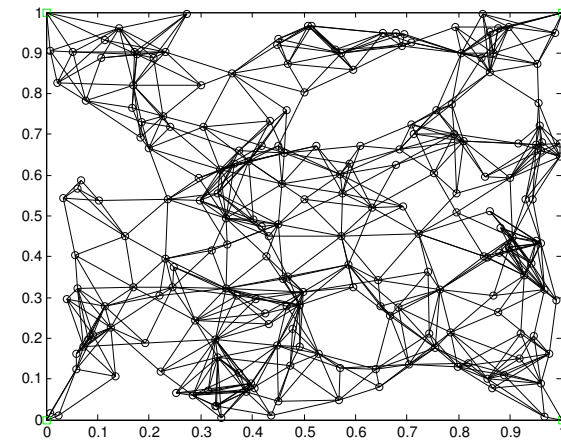
the true location x^p of sensor p
○—○ : the edge (x^p, x^q) with a given exact distance

$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$
anchors at the corner of $[0, 1]^2$, $\rho = 0.15$.



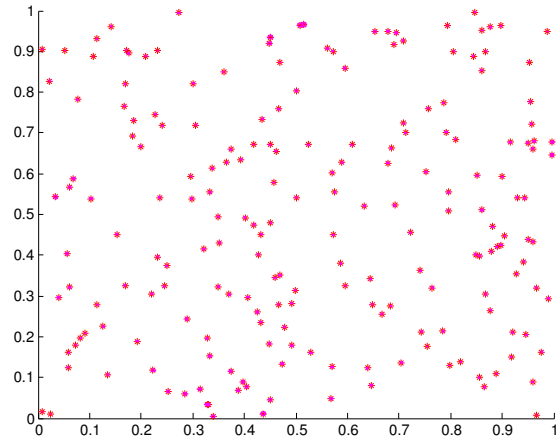
* : c^p = a computed location of sensor p .
the true location x^p of sensor p is within $(\gamma^p)^{1/2} \leq 0.04$ from c^p

$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$
anchors at the corner of $[0, 1]^2$, $\rho = 0.15$.



the true location x^p of sensor p
○—○ : the edge (x^p, x^q) with a given exact distance

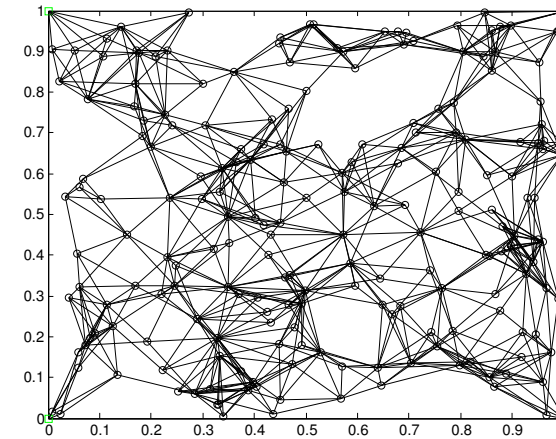
$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$
anchors at the corner of $[0, 1]^2$, $\rho = 0.16$.



* : c^p = a computed location of sensor p .
the true location x^p of sensor p is within $(\gamma^p)^{1/2} \leq 6.0e-3$ from c^p

.. p.34/37

$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$
anchors at the corner of $[0, 1]^2$, $\rho = 0.16$.



the true location x^p of sensor p
○—○ : the edge (x^p, x^q) with a given exact distance

.. p.35/37

Outline

- 1 Problem and Some Formulations
- 2 Theory: Lifting and SDP Relaxation
- 3 Numerical Results
- 4 Applications to the Sensor Network Localization Problem
- 5 Concluding Remarks

.. p.36/37

Concluding Remarks

- We can apply the proposed method to sensor network localization problems with **inexact distance involving measurement error**, but the results are not sharp.
- Polynomial optimization problems with a 0-1 variable x to determine whether $x = 0$ or $x = 1$.

.. p.37/37