

***D*-MODULE STRUCTURE OF LOCAL COHOMOLOGY MODULES OF TORIC ALGEBRAS**

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ABSTRACT. Let S be a toric algebra over a field \mathbb{K} of characteristic 0 and let I be a monomial ideal of S . We show that the local cohomology modules $H_I^i(S)$ are of finite length over the ring of differential operators $D(S; \mathbb{K})$, generalizing the classical case of a polynomial algebra S . As an application, we compute the characteristic cycles of some local cohomology modules.

1. INTRODUCTION

Lyubeznik [Lyu93] introduced an approach of studying local cohomology modules using the theory of D -modules. He obtained many finiteness properties of the local cohomology modules $H_I^i(R)$ when R is a regular ring containing a field of characteristic 0. For example, taking advantage of holonomicity of $H_I^i(R)$ as a D -module, he showed that for any maximal ideal \mathfrak{m} the number of associated prime ideals of $H_I^i(R)$ contained in \mathfrak{m} is finite and that all Bass numbers of $H_I^i(R)$ are finite. When R is a regular local ring of positive characteristic, analogous results were obtained by Huneke and Sharp using the Frobenius functor [HS93]. When R is not regular, the situation is more subtle. There are characteristic-free examples where $H_I^i(R)$ have infinitely many associated primes [SS04]. Also, an example by Hartshorne [Har70] shows that in general the Bass numbers can be infinite (see Example 3.9).

After [Lyu93], there have been several studies on the finiteness properties of R_x and of $H_I^i(R)$ as D -modules for a regular ring R , among them [Bøg95], [Bøg02], [Lyu97], [ÁMBL05]. The first D -finiteness result of R_x for a singular ring R is due to Takagi and Takahashi [TT08] which says R_x is generated by x^{-1} over D when R is a Noetherian graded ring with finite F-representation type. In particular, their theorem applies to the case where R is a normal toric algebra over a perfect field of positive characteristic.

In the present article, we study finiteness properties of the localizations S_f and the local cohomology modules $H_I^i(S)$ as D -modules, where S is a toric algebra (not necessarily normal) over a field \mathbb{K} of characteristic 0, f is a monomial element and I is a monomial ideal of S . In this case, the ring of differential operators $D(S) := D(S; \mathbb{K})$ is much more complicated than the case where S is regular. Using the natural grading of $D(S)$ introduced by Jones [Jon94] and Musson [Mus94], Saito and Traves [ST01, ST04] gave a detailed description of $D(S)$. Based on their results, we prove that any localization $S_f = S[f^{-1}]$ of S is generated by f^{-1} over $D(S)$

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(Theorem 3.1). This implies immediately that $H_I^i(S)$ is $D(S)$ -finitely generated if $D(S)$ is left Noetherian. Unfortunately, $D(S)$ is not always left Noetherian [ST09]. Nonetheless, we can show that $H_I^i(S)$ is actually of finite length as a $D(S)$ -module (Theorem 3.3). In view of Hartshorne's example (Example 3.9), this result is quite surprising.

As an application, we compute the characteristic cycles of some local cohomology modules $H_I^i(S)$. Characteristic cycles are formal sums of subvarieties (counted with multiplicities) of the characteristic variety of a D -module M . Here, the characteristic variety $\text{Ch}(M)$ is the support of the associated graded module $\text{gr } M$ in the spectrum $\text{Spec}(\text{gr } D(S))$ of the associated graded ring $\text{gr } D(S)$. When S is a polynomial algebra, one can explicitly compute the Bass numbers and the associated primes of $H_I^i(S)$ from its characteristic cycles [ÁM04]. The cohomological dimension of I and the Lyubeznik numbers can also be computed from them [ÁM00]. Through our finiteness results of $H_I^i(S)$, we are able to compute the characteristic cycles of some local cohomology modules. We will show that for normal toric algebras S (in fact, for a more general class of toric algebras) the characteristic variety $\text{Ch}(H_m^{\dim S}(S))$ of the top local cohomology with maximal support is abstractly isomorphic to the ambient toric variety $\text{Spec}(S)$ (Theorem 4.13).

In section 2, we briefly recall the notions of local cohomology, toric algebras and the ring of differential operators of a commutative algebra over a field. We also describe the structure of rings of differential operators over toric algebras following the notation in [ST01] and [ST04]. In section 3, our main results on the finiteness properties mentioned above are presented. Also, we relate our finiteness results to the notion of sector partition introduced in [SS90] and [MM06]. Some discussions on $\text{gr } D(S)$ and the computations of characteristic cycles are in section 4. As suggested by the referee, some relations between our results in section 4 and the recent work of Saito [Sai10] are discussed (see Remarks 4.7, 4.14).

2. PRELIMINARIES

2.1. Local cohomology. General facts regarding local cohomology can be found in [ILL+07] or [BS98]. Here, we only recall some basics.

Let R be a Noetherian commutative ring, M an R -module, and I an ideal of R . Define $\Gamma_I(M) := \varinjlim \text{Hom}_R(R/I^k, M)$. Then Γ_I is a left exact R -linear covariant functor and the i -th local cohomology functor H_I^i is defined to be its i -th right derived functor. We call $H_I^i(M)$ the i -th local cohomology module of M supported at the ideal I . If I is generated by f_1, \dots, f_t , then $H_I^i(M)$ is the i -th cohomology of the Čech complex

$$0 \rightarrow M \rightarrow \bigoplus_{i=1}^t M_{f_i} \rightarrow \bigoplus_{1 \leq i < j \leq t} M_{f_i f_j} \rightarrow \cdots \rightarrow M_{f_1 \cdots f_t} \rightarrow 0.$$

2.2. Toric algebras. We introduce some notation for later use. For more information on toric algebras, the reader is referred to [Ful93], [MS05] or [ILL+07].

Let A be a $d \times n$ integer matrix with columns a_1, \dots, a_n . Assume $\mathbb{Z}A = \mathbb{Z}^d$. For a field \mathbb{K} , the semigroup subring $S_{A, \mathbb{K}} := \mathbb{K}[\mathbb{N}A] = \mathbb{K}[t^{a_1}, \dots, t^{a_n}]$ of the Laurent polynomial ring $\mathbb{K}[\mathbb{Z}^d] = \mathbb{K}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ is called the toric algebra associated to the matrix A . Denote $\widetilde{\mathbb{N}A} := \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d$ as the saturation of $\mathbb{N}A$. Then $\widetilde{S_{A, \mathbb{K}}} := \mathbb{K}[\widetilde{\mathbb{N}A}]$ is the normalization of $S_{A, \mathbb{K}}$.

2.3. Rings of differential operators. For a commutative algebra R over a field \mathbb{K} , set $D_0(R; \mathbb{K}) := R$, and for $i > 0$,

$$D_i(R; \mathbb{K}) := \{f \in \text{Hom}_{\mathbb{K}}(R, R) \mid [f, r] \in D_{i-1}(R; \mathbb{K}) \text{ for all } r \in R\}.$$

Then the ring of differential operators is defined to be

$$D(R; \mathbb{K}) := \bigcup_j D_j(R; \mathbb{K}).$$

When R is a polynomial ring over a field \mathbb{K} of characteristic 0, $D(R; \mathbb{K})$ is the usual Weyl algebra. In this paper, a module over $D(R; \mathbb{K})$ means a left $D(R; \mathbb{K})$ -module.

Lemma 2.1. *If M is a $D(R; \mathbb{K})$ -module and $f \in R$, then the R -module structure on M_f extends uniquely to a $D(R; \mathbb{K})$ -module structure such that the natural map $M \rightarrow M_f$ is a $D(R; \mathbb{K})$ -module homomorphism. In particular, via the Čech complex $H_i^{\check{C}}(M)$ has a natural $D(R; \mathbb{K})$ -module structure.*

Proof. See [Lyu00], Example (b). □

When R is a regular algebra over a field \mathbb{K} of characteristic 0, $D(R; \mathbb{K})$ is well understood (see e.g. [Bjö79]). In this case, the local cohomology modules $H_i^{\check{C}}(R)$ are holonomic as $D(R; \mathbb{K})$ -modules and hence are of finite length (see [Lyu93]). This essential property enables Lyubeznik to achieve many finiteness results of the local cohomology modules.

Unfortunately, $D(R; \mathbb{K})$ does not behave well when R is singular; we don't have a notion of holonomicity in this case. This complicates the study of $H_i^{\check{C}}(R)$ via the theory of D -modules. On the bright side, when $R = S_{A, \mathbb{K}}$ is a toric algebra over an algebraically closed field \mathbb{K} of characteristic 0, there is a nice combinatorial structure for $D(R; \mathbb{K})$ which we will present in the next subsection. Our finiteness results about local cohomology modules substantially rely on this structure.

2.4. Rings of differential operators over toric algebras. In the rest of this paper, we denote $S_A := S_{A, \mathbb{K}}$, where \mathbb{K} is an algebraically closed field of characteristic 0. Following [ST01], the noncommutative ring $D_A := D(S_A, \mathbb{K})$ can be described as a \mathbb{Z}^d -graded subring of

$$D(\mathbb{K}[\mathbb{Z}^d]; \mathbb{K}) = \mathbb{K}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \langle \partial_1, \dots, \partial_d \rangle,$$

where $[\partial_i, t_j] = \delta_{ij}$, $[\partial_i, t_j^{-1}] = -\delta_{ij} t_j^{-2}$ and the other pairs of variables commute. More precisely, with the notation $\theta_i := t_i \partial_i$, one has

$$D_A = \bigoplus_{a \in \mathbb{Z}^d} t^a \mathbb{I}(\Omega(a)),$$

where $\Omega(a) = \mathbb{N}A \setminus (-a + \mathbb{N}A)$ and $\mathbb{I}(\Omega(a))$ is the vanishing ideal of $\Omega(a)$ in $\mathbb{K}[\theta_1, \dots, \theta_d]$.

3. FINITENESS PROPERTIES OF $H_i^{\check{C}}(S_A)$

In this section, \mathcal{F} will be denoted to be the set of all facets of $\mathbb{R}_{\geq 0}A$. For a face τ of $\mathbb{R}_{\geq 0}A$, we write $\mathbb{N}(A \cap \tau) := \mathbb{N}A \cap \mathbb{R}\tau$ and denote $\mathbb{Z}(A \cap \tau)$ as the group generated by $\mathbb{N}(A \cap \tau)$.

We recall some notation in [ST01], which are crucial to the proofs of Theorems 3.1 and 3.3.

For $a \in \mathbb{Z}^d$ and τ a face of $\mathbb{R}_{\geq 0}A$, define

$$E_\tau(a) := \{l \in \mathbb{C}(A \cap \tau) \mid a - l \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)\} / \mathbb{Z}(A \cap \tau).$$

Notice that

$$E_\tau(a) \subseteq [\mathbb{Z}^d \cap \mathbb{C}(A \cap \tau)] / \mathbb{Z}(A \cap \tau) = \mathbb{Z}(\widetilde{A \cap \tau}) / \mathbb{Z}(A \cap \tau).$$

Here $\mathbb{Z}(\widetilde{A \cap \tau}) := \mathbb{C}(A \cap \tau) \cap \mathbb{Z}^d$ is the saturation of $\mathbb{Z}(A \cap \tau)$, so each $E_\tau(a)$ is a finite set. Consider the ordering \leq on \mathbb{Z}^d defined by

$$[a \leq b] \iff [E_\tau(a) \subseteq E_\tau(b)] \text{ for all faces } \tau \text{ of } \mathbb{R}_{\geq 0}A.$$

This ordering induces an equivalence relation on \mathbb{Z}^d by

$$[a \sim b] \iff [a \leq b \text{ and } b \leq a].$$

Now we are ready for the first main theorem.

Theorem 3.1. $S_A[f^{-1}] = D_A \cdot f^{-1}$ is cyclic as a left D_A -module for any monomial $f \in S_A$.

Proof. It's clear that $S_A[f^{-1}] \supseteq D_A \cdot f^{-1}$. Conversely, write $f = t^b$ for some $b \in \mathbb{N}A$ (we may assume $b \neq 0$). Since $S_A[f^{-1}] \subseteq D_A \cdot \{t^{-mb} \mid m \in \mathbb{N}\}$, it suffices to show that $t^{-mb} \in D_A \cdot t^{-b}$ for all $m \in \mathbb{N}$. By Proposition 4.1.5(1) in [ST01], it is enough to prove that $-b \sim -mb$. Indeed, we show that, for any face σ , $E_\sigma(-b) = E_\sigma(-mb)$ as follows:

- (1) Let σ be a facet, and suppose $b \notin \sigma$. Then $F_\sigma(-b) < 0$, and hence $F_\sigma(-mb) < 0$. So $E_\sigma(-b) = E_\sigma(-mb) = \emptyset$.
- (2) Let τ be a face and suppose that $b \notin \tau$. Then there exists a facet σ containing τ with $b \notin \sigma$. Hence $E_\sigma(-b) = E_\sigma(-mb) = \emptyset$ by (1), and thus $E_\tau(-b) = E_\tau(-mb) = \emptyset$.
- (3) Let τ be a face and suppose $b \in \tau$. Then $b \in \mathbb{N}A \cap \tau = \mathbb{N}(A \cap \tau)$. Hence, $\pm b \in \mathbb{Z}(A \cap \tau)$, and thus $E_\tau(-b) = E_\tau(-mb)$ by definition. \square

Remark 3.2. Via the Čech complex, it follows immediately from Theorem 3.1 that $H_i^i(S_A)$ is finitely generated over D_A if D_A is left Noetherian. The left Noetherianity of D_A was studied by Saito and Takahashi [ST09]. They proved that D_A is left Noetherian if S_A satisfies Serre's condition (S_2) . Serre's condition is, by [Ish88], equivalent to

$$S_A = \bigcap_{\tau: \text{ facets}} \mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \tau)].$$

Saito and Takahashi also gave a necessary condition (on S_A) for D_A to be left Noetherian. However, D_A is not always left Noetherian.

Nonetheless, we have

Theorem 3.3. For any i and any monomial ideal I , $H_i^i(S_A)$ is of finite length as a D_A -module.

Proof. In view of the Čech complex, since any localization $S_A[f^{-1}]$ (with monomial f) is a D_A -submodule of $\mathbb{K}[\mathbb{Z}^d]$, it suffices to show that $\mathbb{K}[\mathbb{Z}^d]$ has a composition series.

Consider the notation in the beginning of this section. For $a, b \in \mathbb{Z}^d$, we will write $a < b$ if $a \leq b$ but $a \not\approx b$. Then, for each $a \in \mathbb{Z}^d$, $\bigoplus_{b \geq a} \mathbb{K}t^b$ is generated by t^a as a D_A -module. Moreover,

$$\frac{\bigoplus_{b \geq a} \mathbb{K}t^b}{\bigoplus_{b > a} \mathbb{K}t^b} \cong \bigoplus_{b \in [a]} \mathbb{K}t^b$$

is, by Theorem 4.1.6 in [ST01], a simple D_A -module where $[a] = \{b \in \mathbb{Z}^d \mid b \sim a\}$. Since there are only finitely many faces and since each $E_\tau(a)$ is contained in $\mathbb{Z}(\widetilde{A \cap \tau})/\mathbb{Z}(A \cap \tau)$, there are finitely many equivalence classes determined by \sim ; we denote them $[\alpha_1], [\alpha_2], \dots, [\alpha_k]$. We may rearrange the order so that, for any pair $i < j$, either $\alpha_i > \alpha_j$ or α_i and α_j are incomparable. Denote $T_a := \bigoplus_{b \geq a} \mathbb{K}t^b$. Then the filtration

$$0 \subsetneq T_{\alpha_1} \subsetneq \dots \subsetneq \Sigma_{l=1}^i T_{\alpha_l} \subsetneq \dots \subsetneq \Sigma_{l=1}^k T_{\alpha_l} = \mathbb{K}[\mathbb{Z}^d]$$

is a composition series of D_A -submodules of $\mathbb{K}[\mathbb{Z}^d]$. □

Example 3.4. For 1-dimensional S_A , the composition series of $\mathbb{K}[\mathbb{Z}]$ is easy to describe. In this case, $\mathbb{R}_{\geq 0}A$ has two faces, 0 and $\sigma = \mathbb{R}_{\geq 0}A$. For $a \in \mathbb{Z}$,

$$E_0(a) = \{\ell \in \{0\} \mid a - \ell \in \mathbb{N}A\}/\{0\} \text{ and} \\ E_\sigma(a) = \{\ell \in \mathbb{Z} \mid a - \ell \in \mathbb{Z}\}/\mathbb{Z}.$$

Thus $E_0(a) = \{0\}$ if $a \in \mathbb{N}A$, $E_0(a) = \emptyset$ if $a \notin \mathbb{N}A$, and $E_\sigma(a) = \{0\}$ for all $a \in \mathbb{Z}$. Therefore, $[0]$ and $[-1]$ are the two equivalence classes determined by \sim , and we have the composition series

$$0 \subsetneq \mathbb{K}[\mathbb{N}A] = T_0 \subsetneq T_{-1} = \mathbb{K}[\mathbb{Z}].$$

Remark 3.5. Theorem 3.1 and Theorem 3.3 also hold for any field \mathbb{K} with characteristic 0 by the isomorphism

$$H_i^i(S_{A, \mathbb{K}}) \otimes \overline{\mathbb{K}} \cong H_i^i(S_{A, \overline{\mathbb{K}}}).$$

Remark 3.6. Suppose $I = \mathfrak{m}$, the maximal graded ideal of S_A . Here we assume that the semigroup $\mathbb{N}A$ is pointed, so that 0 is the only invertible element.

- (1) Recall that $H_m^i(S_A)$ can be computed as the i -th cohomology of the Ishida complex [Ish88] or [ILL+07]. Therefore, $H_m^1(S_A)$ is finitely generated as an S_A -module. Indeed, it suffices to observe that

$$H_m^1(S_A) \leftarrow \bigcap_{\sigma: \text{ rays}} \mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \sigma)] \subseteq \bigcap_{\tau: \text{ facets}} \mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \tau)] \subseteq \widetilde{S}_A$$

and the fact that \widetilde{S}_A is finite over S_A . Moreover, $H_m^d(S_A)$ is cyclic as a left D_A -module. This is because the d -th module in the Ishida complex is $\mathbb{K}[\mathbb{Z}^d]$, which is cyclic by Theorem 3.1.

- (2) In general, Schäfer and Schenzel [SS90] showed that there is a partition of \mathbb{Z}^d with respect to which $H_m^i(S_A)$ can be written as a finite direct sum of \mathbb{K} -vector spaces. This decomposition coincides with the sector partition appearing in [MM06] (see also [HM05] for a more general notion of sector partition). More precisely, let $\text{Conv}(A)$ be the set of all faces of $\mathbb{R}_{\geq 0}A$, and for any filter (cocomplex) ∇ of $\text{Conv}(A)$, denote

$$P_\nabla = \bigcap_{A \cap \tau \in \nabla} [\mathbb{N}A + \mathbb{Z}(A \cap \tau)] \setminus \bigcup_{A \cap \tau \notin \nabla} [\mathbb{N}A + \mathbb{Z}(A \cap \tau)].$$

Then the P_{∇} 's form a partition (sector partition) of \mathbb{Z}^d and

$$H_m^i(S_A) = \bigoplus_{\nabla} \mathbb{K}[P_{\nabla}] \otimes_{\mathbb{K}} H^i(\text{Conv}(A), \text{Conv}(A) \setminus \nabla; \mathbb{K}).$$

On the other hand, for $a \in \mathbb{Z}^d$ denote

$$\nabla(a) := \{\text{face } \tau \text{ of } \mathbb{R}_{\geq 0}A \mid a \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)\}$$

and consider the equivalence relation $a \equiv a' \Leftrightarrow \nabla(a) = \nabla(a')$. Then $P_{\nabla(a)}$ is the equivalence class containing a . Notice that P_{∇} could be empty and that $[a \in P_{\nabla}] \Leftrightarrow [P_{\nabla} = P_{\nabla(a)}]$.

Theorem 6 in [MM06] shows that the partition determined by the equivalence relation \sim in the proof of Theorem 3.3 is finer than the sector partition determined by \equiv .

Notice that each $\mathbb{K}[P_{\nabla}]$ is naturally a left \mathbb{Z}^d -graded D_A -module because each $\mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \tau)]$ is as well. If S_A is normal, the D_A -module $\mathbb{K}[P_{\nabla}]$ is simple. In fact, we have

Theorem 3.7. *If S_A is normal and I is a monomial ideal in S_A , then every simple subquotient of $H_I^i(S_A)$ is of the form $\mathbb{K}[P_{\nabla}]$ coming from the sector partition.*

Proof. By Theorem 3.3 and Remark 3.6(2), we only have to show that \sim and \equiv define the same equivalence relation on \mathbb{Z}^d . Note that the normality of S_A implies that

$$\begin{aligned} E_{\tau}(a) &= \{0\} \text{ if } a \in \mathbb{N}A + \mathbb{Z}(A \cap \tau) \text{ and} \\ E_{\tau}(a) &= \emptyset \text{ if } a \notin \mathbb{N}A + \mathbb{Z}(A \cap \tau). \end{aligned}$$

So we have

$$\begin{aligned} a &\sim b \\ \Leftrightarrow E_{\tau}(a) &= E_{\tau}(b) \text{ for all } \tau \\ \Leftrightarrow E_{\tau}(a) &= \{0\} \text{ if and only if } E_{\tau}(b) = \{0\} \\ \Leftrightarrow a \in \mathbb{N}A + \mathbb{Z}(A \cap \tau) &\text{ if and only if } b \in \mathbb{N}A + \mathbb{Z}(A \cap \tau) \\ \Leftrightarrow a &\equiv b. \end{aligned}$$

Therefore, the simple subquotients of $\mathbb{K}[\mathbb{Z}^d]$ are precisely the D_A -modules

$$\mathbb{K}[P_{\nabla}] = \bigoplus_{b \in [a]} \mathbb{K}t^b \cong \frac{\bigoplus_{b \geq a} \mathbb{K}t^b}{\bigoplus_{b > a} \mathbb{K}t^b}. \quad \square$$

Example 3.8. Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Then $S_A = \mathbb{K}[t, ts, ts^2]$ is a 2-dimensional normal toric algebra. Let $I = (ts)$ be the ideal of S_A generated by ts . We shall describe a composition series of $H_I^1(S_A)$. By Čech complex, $H_I^1(S_A) = \mathbb{K}[\mathbb{Z}^d]/S_A$. Following the notation in the proof of Theorem 3.3 and Remark 3.6, let

$$\begin{aligned} \nabla_0 &= \{0, \sigma_1, \sigma_2, \mathbb{R}_{\geq 0}A\}, \nabla_1 = \{\sigma_1, \mathbb{R}_{\geq 0}A\}, \nabla_2 = \{\sigma_2, \mathbb{R}_{\geq 0}A\}, \\ \nabla_{12} &= \{\sigma_1, \sigma_2, \mathbb{R}_{\geq 0}A\}, \nabla_A = \{\mathbb{R}_{\geq 0}A\}, \end{aligned}$$

where $\sigma_1 = \mathbb{R}_{\geq 0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\sigma_2 = \mathbb{R}_{\geq 0} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then

$$\begin{aligned} P_{\nabla_0} &= P_{\nabla(a_0)} = \mathbb{N}A, \\ P_{\nabla_1} &= P_{\nabla(-a_1)} = [\mathbb{N}A + \mathbb{Z}(A \cap \sigma_1)] \setminus [\mathbb{N}A + \mathbb{Z}(A \cap \sigma_2)], \\ P_{\nabla_2} &= P_{\nabla(-a_2)} = [\mathbb{N}A + \mathbb{Z}(A \cap \sigma_2)] \setminus [\mathbb{N}A + \mathbb{Z}(A \cap \sigma_1)], \\ P_{\nabla_{12}} &= \emptyset, \text{ and} \\ P_{\nabla_A} &= P_{\nabla(-a_3)} = \mathbb{Z}^2 \setminus [(\mathbb{N}A + \mathbb{Z}(A \cap \sigma_1)) \cup (\mathbb{N}A + \mathbb{Z}(A \cap \sigma_2))], \end{aligned}$$

where $a_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $a_i, i = 1, 2, 3$, is the i -th column of A . In terms of notation in Theorem 3.3,

$$\begin{aligned} T_{a_0} &= \mathbb{K}[P_{\nabla_0}] = S_A, \\ T_{-a_1} &= \mathbb{K}[P_{\nabla_0}] \oplus \mathbb{K}[P_{\nabla_1}], \\ T_{-a_2} &= \mathbb{K}[P_{\nabla_0}] \oplus \mathbb{K}[P_{\nabla_2}], \\ T_{-a_3} &= \mathbb{K}[\mathbb{Z}^2] = \mathbb{K}[P_{\nabla_0}] \oplus \mathbb{K}[P_{\nabla_1}] \oplus \mathbb{K}[P_{\nabla_2}] \oplus \mathbb{K}[P_{\nabla_A}]. \end{aligned}$$

So $0 \subset T_{a_0} \subset T_{-a_1} \subset T_{-a_1} + T_{-a_2} \subset T_{-a_3} = \mathbb{K}[\mathbb{Z}^2]$ is a composition series of $\mathbb{K}[\mathbb{Z}^2]$. Quotienting out S_A , we obtain a composition series of $H_I^1(S_A)$:

$$0 \subset \mathbb{K}[P_{\nabla_1}] \subset \mathbb{K}[P_{\nabla_1}] \oplus \mathbb{K}[P_{\nabla_2}] \subset \mathbb{K}[P_{\nabla_1}] \oplus \mathbb{K}[P_{\nabla_2}] \oplus \mathbb{K}[P_{\nabla_A}].$$

Example 3.9. This example is essentially due to Hartshorne [Har70]. We adopt its combinatorial description which can be found in [ILL+07] or [MS05].

Consider $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then $S_A = \mathbb{K}[r, rs, rt, rst]$ is a normal toric algebra. Consider the ideal $I = (r, rs)$ of S_A . Then the socle $\text{Hom}_{S_A}(S_A/\mathfrak{m}, H_I^2(S_A))$ is infinite dimensional, where $\mathfrak{m} = (r, rs, rt, rst)$ is the maximal graded ideal of S_A . However, according to Theorem 3.3 $H_I^2(S_A)$ is of finite length over D_A .

In fact, $H_I^2(S_A)$ is D_A -simple. To see this, using the notation in Remark 3.6 we consider the filter $\nabla = \{\sigma_{12}, \mathbb{R}_{\geq 0}A\}$, where $\sigma_{12} = \mathbb{R}_{\geq 0} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is a facet of $\mathbb{R}_{\geq 0}A$. We claim that

$$P_{\nabla} = [\mathbb{N}A + \mathbb{Z}(A \cap \sigma_{12})] \setminus [(\mathbb{N}A + \mathbb{Z}a_1) \cup (\mathbb{N}A + \mathbb{Z}a_2)],$$

where $a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $a_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, in view of the Čech complex we have the isomorphism

$$H_I^2(S_A) \cong \mathbb{K}[P_{\nabla}],$$

which is D_A -simple by Theorem 3.7.

The claim is equivalent to the equality

$$\begin{aligned} &[\mathbb{N}A + \mathbb{Z}(A \cap \sigma_{12})] \cap \left[\bigcup_{\sigma = \sigma_{13}, \sigma_{24}, \sigma_{34}} (\mathbb{N}A + \mathbb{Z}(A \cap \sigma)) \right] \\ &= [\mathbb{N}A + \mathbb{Z}(A \cap \sigma_{12})] \cap [(\mathbb{N}A + \mathbb{Z}a_1) \cup (\mathbb{N}A + \mathbb{Z}a_2)], \end{aligned}$$

where $\sigma_{13} = \mathbb{R}_{\geq 0} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\sigma_{24} = \mathbb{R}_{\geq 0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $\sigma_{34} = \mathbb{R}_{\geq 0} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This equality can be verified by the following data:

$$\begin{aligned} \mathbb{N}A + \mathbb{Z}a_1 &= \{^t(x, y, z) \in \mathbb{Z}^3 \mid y \geq 0 \text{ and } z \geq 0\}, \\ \mathbb{N}A + \mathbb{Z}a_2 &= \{^t(x, y, z) \in \mathbb{Z}^3 \mid x \geq y \text{ and } z \geq 0\}, \\ \mathbb{N}A + \mathbb{Z}(A \cap \sigma_{12}) &= \{^t(x, y, z) \in \mathbb{Z}^3 \mid z \geq 0\}, \\ \mathbb{N}A + \mathbb{Z}(A \cap \sigma_{13}) &= \{^t(x, y, z) \in \mathbb{Z}^3 \mid y \geq 0\}, \\ \mathbb{N}A + \mathbb{Z}(A \cap \sigma_{24}) &= \{^t(x, y, z) \in \mathbb{Z}^3 \mid x \geq y\}, \\ \mathbb{N}A + \mathbb{Z}(A \cap \sigma_{34}) &= \{^t(x, y, z) \in \mathbb{Z}^3 \mid x \geq z\}. \end{aligned}$$

Remark 3.10. Helm and Miller [HM03] studied the Bass numbers of local cohomology modules over toric algebras. As a generalization of Hartshorne’s example, their main result ([HM03], Theorem 7.1) implies that for a Gorenstein normal toric algebra S_A , $\mathbb{N}A$ is not simplicial if and only if there exists an $\mathbb{N}A$ -graded prime \mathfrak{p} of dimension 2 such that $H_{\mathfrak{p}}^{d-1}(S_A)$ has infinite-dimensional socle.

4. ASSOCIATED GRADED RINGS $\text{gr } D_A$ AND CHARACTERISTIC CYCLES

4.1. **Associated graded rings $\text{gr } D_A$.** Let R be a \mathbb{K} -algebra as in subsection 2.3. The definition of $D(R; \mathbb{K})$ gives an order filtration of $D(R; \mathbb{K})$:

$$0 \subseteq R = D_0(R; \mathbb{K}) \subseteq D_1(R; \mathbb{K}) \subseteq D_2(R; \mathbb{K}) \subseteq \cdots .$$

Define the associated graded ring of $D(R; \mathbb{K})$ to be

$$\text{gr } D(R; \mathbb{K}) := D_0 \oplus (D_1/D_0) \oplus (D_2/D_1) \oplus \cdots ,$$

where $D_i := D_i(R; \mathbb{K})$. From the definition of $D(R; \mathbb{K})$, $\text{gr } D(R; \mathbb{K})$ is a commutative R -algebra and we have the natural embedding $R \hookrightarrow \text{gr } D(R; \mathbb{K})$. For example, if $R = \mathbb{K}[t_1, \dots, t_d]$ is a polynomial algebra over \mathbb{K} , then $D(R; \mathbb{K}) = \mathbb{K}[t_1, \dots, t_d][\partial_1, \dots, \partial_d]$ is the Weyl algebra. The associated graded ring $\text{gr } D(R; \mathbb{K}) = \mathbb{K}[t_1, \dots, t_d, \xi_1, \dots, \xi_d]$ is a $2d$ -dimensional polynomial algebra over \mathbb{K} , where ξ_i is the image of ∂_i in the associated graded ring $\text{gr } D(R; \mathbb{K})$. In what follows, we will use the description

$$\text{gr } D(\mathbb{K}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]; \mathbb{K}) = \mathbb{K}[t_1^{\pm 1}, \dots, t_d^{\pm 1}, \Theta_1, \dots, \Theta_d],$$

where $\Theta_i = t_i \xi_i$ is the image of $\theta_i = t_i \partial_i$ in the associated graded ring $\text{gr } D(R; \mathbb{K})$.

When R is a regular algebra over \mathbb{K} , $\text{Spec}(\text{gr } D(R; \mathbb{K}))$ can be identified as the cotangent bundle of the variety $\text{Spec } R$ with the projection

$$\pi : \text{Spec}(\text{gr } D(R; \mathbb{K})) \rightarrow \text{Spec } R$$

induced by the embedding $R \hookrightarrow \text{gr } D(R; \mathbb{K})$. The fiber of π over a closed point of $\text{Spec } R$ is the cotangent space over that point which is isomorphic to the affine space $\mathbb{K}^{\dim R}$.

In this subsection, we discuss the map π for a certain class of toric algebras. We shall see that in some cases the fibers of π behave nicely (Theorems 4.5, 4.8). We also give an example (Example 4.6) of a more complicated nature.

To begin with, consider the natural order filtration of D_A inherited from that of $D(\mathbb{K}[\mathbb{Z}^d])$. With respect to this filtration, one can regard $\text{gr } D_A$ as a commutative subalgebra of $\text{gr } D(\mathbb{K}[\mathbb{Z}^d]) = \mathbb{K}[t_1^{\pm 1}, \dots, t_d^{\pm 1}, \Theta_1, \dots, \Theta_d]$. When $\text{gr } D_A$ is finitely generated over \mathbb{K} , Musson showed that it has dimension $2d$ [Mus87]. Saito and

Traves proved that $\text{gr } D_A$ is finitely generated over \mathbb{K} if and only if the semigroup $\mathbb{N}A$ is scored [ST04]. By definition, a semigroup $\mathbb{N}A$ is scored if

$$\mathbb{N}A = \bigcap_{\sigma: \text{facet}} \{a \in \mathbb{Z}^d \mid F_\sigma(a) \in F_\sigma(\mathbb{N}A)\}$$

or, equivalently, $\widehat{\mathbb{N}A} \setminus \mathbb{N}A$ is a union of finitely many hyperplane sections parallel to some facets of $\mathbb{R}_{\geq 0}A$. The scored condition implies Serre’s condition (S_2) (see Remark 3.2).

We should remark that if S_A is normal, then $\text{gr } D_A$ is Gorenstein ([Mus87], Theorem D). In general, even for a 1-dimensional semigroup ring (which is always scored), the associated graded ring can have bad singularities. In fact, we have the following.

Proposition 4.1. *If S_A is a 1-dimensional toric algebra which is not normal, then $\text{gr } D_A$ is not Cohen-Macaulay.*

Proof. Using the formula in Lemma 4.3, we see that $\text{gr } D_A$ is again a toric algebra over \mathbb{K} . Indeed, notice that 0 is the only facet and that $n_{0,w} = |\Omega(w)|$. Note also that $|\Omega(-w)| = w + |\Omega(w)|$ by Lemma 4.4. So by Lemma 4.3,

$$\text{gr } D_A = \mathbb{K} \left[t\xi, t^{|\Omega(w)|}\xi^{|\Omega(-w)|} \mid |w| \in \{a_1, \dots, a_n\} \cup \text{Hole}(\mathbb{N}A) \right].$$

Therefore $\text{gr } D_A$ is a two-dimensional toric algebra over \mathbb{K} .

We claim that

$$(4.1) \quad \dim_{\mathbb{K}} \frac{\mathbb{K}[t, \xi]}{\text{gr } D_A} < \infty.$$

Take ℓ to be the maximal number of $2|\Omega(-w)|$ for $w \in \text{Hole}(\mathbb{N}A)$. To prove (4.1), it is enough to show that $t^u\xi^v \in \text{gr } D_A$ for all pairs $u, v \in \mathbb{N}$ satisfying $u + v \geq \ell$. Since $[t^u\xi^v \in \text{gr } D_A \Leftrightarrow t^v\xi^u \in \text{gr } D_A]$, by symmetry we may assume $w_0 := u - v \geq 0$.

- If $w_0 \in \text{Hole}(\mathbb{N}A)$, $2u \geq u + v \geq \ell \geq 2|\Omega(-w_0)|$. So $(t\xi)^{u-|\Omega(-w_0)|} \in \text{gr } D_A$, and hence $t^u\xi^v = (t\xi)^{u-|\Omega(-w_0)|} t^{|\Omega(-w_0)|}\xi^{|\Omega(w_0)|} \in \text{gr } D_A$.
- If $w_0 \in \mathbb{N}A$ we have $t^u\xi^v = (t\xi)^v t^{w_0} = (t\xi)^v t^{|\Omega(-w_0)|} \in \text{gr } D_A$.

So the claim is proved. Now, applying the criterion in Remark 3.2 to the toric algebra $\text{gr } D_A$, we see that $\text{gr } D_A$ doesn’t satisfy Serre’s condition (S_2) . Hence, $\text{gr } D_A$ is not Cohen-Macaulay. □

Remark 4.2. The claim (4.1) holds true for any affine curve with injective normalization (see the proof of Theorem 3.12 in [SS88]). In fact, this codimension is known to be the Letzter–Makar-Limanov invariant, which plays an important role in the theory of Calogero-Moser space. For more information, see the work by Berest and Wilson [BW99].

Now, let \mathfrak{m} be the maximal graded ideal of S_A corresponding to the closed point 0 of the toric variety $\text{Spec}(S_A)$. Let $I = \sqrt{\mathfrak{m}\text{gr } D_A}$ be the radical of the extended ideal of \mathfrak{m} under the embedding $S_A \hookrightarrow \text{gr } D_A$. We are going to show that if $\mathbb{N}A$ is simplicial and scored, then $\text{gr } D_A/I$ is isomorphic to S_A as \mathbb{K} -algebras. This implies that the reduced induced structure of the fiber of $\pi : \text{Spec}(\text{gr } D_A) \rightarrow \text{Spec}(S_A)$ over the point 0 is isomorphic to the ambient toric variety.

The following two lemmas are needed:

Lemma 4.3 ([ST04]). *For scored $\mathbb{N}A$,*

$$\text{gr } D_A = \bigoplus_{a \in \mathbb{Z}^d} t^a \mathbb{K}[\Theta_1, \dots, \Theta_d] \cdot P_a \text{ where}$$

$$P_a = \prod_{\sigma \in \mathcal{F}} F_\sigma(\Theta)^{n_{\sigma,a}} \text{ and } n_{\sigma,a} = \#\{F_\sigma(\mathbb{N}A) \setminus [-F_\sigma(a) + F_\sigma(\mathbb{N}A)]\}.$$

Lemma 4.4. *Let $\mathbb{N}A$ be scored. Then for any $a \in \mathbb{Z}^d$ and $\sigma \in \mathcal{F}$,*

$$n_{\sigma,-a} = n_{\sigma,a} + F_\sigma(a).$$

In particular,

- (1) *if σ is a facet with the property that $F_\sigma(a) \leq 0$, then $n_{\sigma,ka} \leq k \cdot n_{\sigma,a}$ for large $k \in \mathbb{N}$, and furthermore $P_a^k = P_{ka} \cdot P$ for some $P \in \mathbb{K}[\Theta]$;*
- (2) *if $F_\sigma(a) \leq 0$ for all $\sigma \in \mathcal{F}$ and $-a \notin \mathbb{N}A$, then $n_{\sigma,ka} < k \cdot n_{\sigma,a}$ for some $\sigma \in \mathcal{F}$ and large $k \in \mathbb{N}$;*
- (3) *if $a \in \mathbb{N}A$, then $n_{\sigma,-a} = F_\sigma(a)$;*
- (4) *if $F_\sigma(a) > 0$, then $n_{\sigma,ka} = 0$ for large $k \in \mathbb{N}$.*

Proof. To prove $n_{\sigma,-a} = n_{\sigma,a} + F_\sigma(a)$ for any $a \in \mathbb{Z}^d$ and $\sigma \in \mathcal{F}$, it's enough to show the case where $F_\sigma(a) > 0$. Set $N = F_\sigma(\mathbb{N}A)$ and $n = F_\sigma(a) > 0$. Then

$$\begin{aligned} n_{\sigma,-a} &= \#\{N \setminus (n + N)\} \\ &= \#\{b \in N \mid b - n \notin N\} \\ &= n + \#\{b \in N \mid b - n \notin N \text{ but } b - kn \in N \text{ for some } k \geq 2\} \\ &= n + \#\{c \in N \mid c + n \notin N\} \\ &= n + \#\{N \setminus (-n + N)\} = n + n_{\sigma,a}. \end{aligned}$$

Note that for the second equality we need the assumption that $\mathbb{N}A$ is scored.

Now, we prove the four additional statements:

- (1) Notice that $n_{\sigma,ka} = kF_\sigma(-a)$ for large k and that $n_{\sigma,a} = n_{\sigma,-a} + F_\sigma(-a)$ where $n_{\sigma,-a} \geq 0$.
- (2) By assumption, $-a$ lies on a hyperplane parallel to some facet, say σ_0 . Then $n_{\sigma_0,-a} > 0$ and hence $n_{\sigma_0,ka} < k \cdot n_{\sigma_0,a}$ by (1).
- (3) $a \in \mathbb{N}A$ implies $n_{\sigma,a} = 0$.
- (4) This follows from the definition. Indeed, since $\mathbb{N}A$ is scored, $(F_\sigma(ka) + \mathbb{N}_0) \subseteq F_\sigma(\mathbb{N}A)$ for large k . Then $F_\sigma(ka) + F_\sigma(\mathbb{N}A) \subseteq F_\sigma(\mathbb{N}A)$, and hence $n_{\sigma,ka} = 0$. □

Theorem 4.5. *If $\mathbb{N}A$ is a simplicial scored semigroup, then*

$$\text{gr } D_A/I = \mathbb{K} \left[\overline{t^{-a_i} \cdot P_{-a_i}}; i = 1, \dots, n \right] \cong S_A,$$

where $I = \sqrt{\mathfrak{m} \text{ gr } D_A}$ and $\overline{t^{-a_i} \cdot P_{-a_i}}$ is the image of $t^{-a_i} \cdot P_{-a_i}$ in $\text{gr } D_A/I$.

Proof. We sketch how the proof of the left equality goes. Let $\mathcal{F} = \{\sigma_1, \dots, \sigma_d\}$ be the set of all facets of $\mathbb{R}_{\geq 0}A$, and let

$$C = -\widetilde{\mathbb{N}A} = \{a \in \mathbb{Z}^d \mid F_\sigma(a) \leq 0 \text{ for all } \sigma \in \mathcal{F}\}.$$

We will prove the left equality in three steps. The first step shows $\Theta_i \in I$ for $i = 1, \dots, d$. The second step shows $t^a \cdot P_a \in I$ for all $a \in \mathbb{Z}^d \setminus C$. Finally, the third step shows that $t^a \cdot P_a \in I$ if $a \in C \setminus (-\mathbb{N}A)$ and that $t^a \cdot P_a$ is a product of some $t^{-a_i} \cdot P_{-a_i}$, $i = 1, \dots, n$, if $a \in C \cap (-\mathbb{N}A \setminus \{0\})$.

- (1) For each $i = 1, \dots, d$, consider the following subset of \mathbb{Z}^d :

$$\{F_{\sigma_i}(\Theta) = -1\} \cap \left[\bigcap_{j \neq i} \{F_{\sigma_j}(\Theta) = 0\} \right].$$

Since $\mathbb{N}A$ is simplicial, this is a one point set for each i , say $\{u_i\}$. Notice that since $t^{-u_i} \in I$, $F_{\sigma_i}^{n_{\sigma_i, u_i}} = P_{u_i} = t^{-u_i} \cdot t^{u_i} P_{u_i} \in I$, where $n_{\sigma_i, u_i} > 0$. Therefore, $F_{\sigma_i} \in I$ for each i . Since $F_{\sigma_1}, \dots, F_{\sigma_d}$ are linearly independent, we conclude that $\Theta_i \in I$ for $i = 1, \dots, d$.

- (2) For $a \in \mathbb{Z}^d \setminus C$, $F_{\sigma}(a) > 0$ for some $\sigma \in \mathcal{F}$. By Lemma 4.4(4), choose k large so that

$$P_{ka} = \prod_{F_{\sigma}(a) < 0} F_{\sigma}^{n_{\sigma, ka}}.$$

Now, consider as in (1) the one-point set

$$\left[\bigcap_{F_{\sigma}(a) < 0} \{F_{\sigma}(\Theta) = F_{\sigma}(ka)\} \right] \cap \left[\bigcap_{F_{\sigma}(a) \geq 0} \{F_{\sigma}(\Theta) = 0\} \right] = \{b\}.$$

We have $P_b = P_{ka}$ and $t^{ka-b} \in I$. By Lemma 4.4(1)

$$(t^a P_a)^k = t^{ka} P_a^k = t^{ka} P_{ka} \cdot P = (t^b P_b) \cdot P \cdot t^{ka-b} \in I.$$

Therefore, $t^a P_a \in I$ as desired.

- (3) Let $a \in C$.

If $-a \notin \mathbb{N}A$, by Lemma 4.4(2) $(t^a P_a)^k = t^{ka} P_{ka} \cdot P$ for some nonconstant $P \in \mathbb{K}[\Theta]$. Since P is a product of some F_{σ} 's, $P \in I$ by (1), and hence $t^a P_a \in I$.

If $-a \in \mathbb{N}A \setminus \{0\}$, write $-a = \sum m_i a_i$. By Lemma 4.4(3),

$$n_{\sigma, a} = F_{\sigma}(-a) = \sum m_i F_{\sigma}(a_i) = \sum m_i n_{\sigma, -a_i},$$

and therefore

$$t^a P_a = t^{\sum m_i (-a_i)} \cdot \prod_{\sigma \in \mathcal{F}} F_{\sigma}^{\sum m_i n_{\sigma, -a_i}} = \prod (t^{-a_i} P_{-a_i})^{m_i}.$$

To complete the proof, we establish the right isomorphism. First, recall that if $R \rightarrow S$ is a homomorphism of commutative rings and Q is a prime ideal in S lying over a prime ideal q of R , then $\dim(S_Q/qS_Q) \geq \text{ht}Q - \text{ht}q$. On the other hand, since $\text{gr} D_A$ is finitely generated as a \mathbb{K} -algebra which is also a domain, each maximal ideal of $\text{gr} D_A$ has height $2d$. (Here, we use the fact that $\dim \text{gr} D_A = 2d$.) Therefore, $\dim(\text{gr} D_A/I) \geq d$. Now, consider the surjection from the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ to $\mathbb{K} \left[\overline{t^{-a_i} \cdot P_{-a_i}}; i = 1, \dots, n \right]$. By Lemma 4.4(3), $P_{-a_i} = \prod_{\sigma \in \mathcal{F}} F_{\sigma}^{F_{\sigma}(a_i)}$. Observe that $\overline{t^{-a_i} \cdot P_{-a_i}}, i = 1, \dots, n$, satisfy the relations in the toric ideal $I_A = \{x^u - x^v \mid Au = Av\}$ (where for $u \in \mathbb{Z}^n$, $x^u := x_1^{u_1} \cdots x_n^{u_n}$).

Hence we have a surjection

$$S_A \cong \mathbb{K}[x_1, \dots, x_n]/I_A \longrightarrow \mathbb{K} \left[\overline{t^{-a_i} \cdot P_{-a_i}}; i = 1, \dots, n \right],$$

which is an isomorphism by comparing the dimensions. □

Example 4.6. Consider

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$S_A = \mathbb{K}[s, t, u, stu^{-1}]$ is a 3-dimensional normal toric algebra which is isomorphic to the toric algebra appearing in Example 3.9. By 4.1, 4.6, and 6.3 in [ST04],

$$\text{gr } D_A = \mathbb{K}[s, t, u, stu^{-1}, \Theta_s, \Theta_t, \Theta_u, s^{-1}\Theta_s(\Theta_s + \Theta_u), t^{-1}\Theta_t(\Theta_t + \Theta_u), s^{-1}t^{-1}u\Theta_s\Theta_t u^{-1}(\Theta_s + \Theta_u)(\Theta_t + \Theta_u), tu^{-1}(\Theta_s + \Theta_u), t^{-1}u\Theta_t, su^{-1}(\Theta_t + \Theta_u), s^{-1}u\Theta_s].$$

Set

$$\begin{aligned} a &= s, b = t, c = u, d = stu^{-1}, e = \Theta_s, f = \Theta_t, g = \Theta_u, h = s^{-1}\Theta_s(\Theta_s + \Theta_u), \\ i &= t^{-1}\Theta_t(\Theta_t + \Theta_u), j = u^{-1}(\Theta_s + \Theta_u)(\Theta_t + \Theta_u), k = s^{-1}t^{-1}u\Theta_s \cdot \Theta_t, \\ l &= tu^{-1}(\Theta_s + \Theta_u), m = t^{-1}u\Theta_t, n = su^{-1}(\Theta_t + \Theta_u), o = s^{-1}u\Theta_s. \end{aligned}$$

Consider the surjection $\phi : \mathbb{K}[a, \dots, o] \rightarrow \text{gr } D_A$. Using Macaulay 2, we see that a primary decomposition of $\sqrt{\mathfrak{m} \text{gr } D_A}$ is the intersection of the two ideals $(o, n, d, a, c, f + g, e, b, fj - il, fi + jm, f^2 + lm, fk + hm, hi - jk, fh - kl)$ and $(m, l, d, a, c, f, e + g, b, gk - io, gi + kn, g^2 + no, gj + hn, hi - jk, gh - jo)$ modulo $\text{Ker } \phi$. Therefore, the fiber $\pi^{-1}(0)$ has two components, each of which is 4-dimensional.

Remark 4.7. The left equality of Theorem 4.5 and Example 4.6 can be achieved alternatively by a result of Saito.

By Proposition 4.14 in [Sai10], $\pi^{-1}(\mathfrak{m}) = \{\mathfrak{P}(\mathfrak{q}, \nu) \mid \nu \cap \mathbb{R}_{\geq 0}A = \{0\}\}$. If $\mathbb{N}A$ is simplicial, then $\pi^{-1}(\mathfrak{m}) = \{\mathfrak{P}(\mathfrak{m}_0, -\mathbb{R}_{\geq 0}A)\}$, which is the left equality of Theorem 4.5. On the other hand, consider Example 4.6. If $\mathfrak{P}(\mathfrak{q}, \nu) \in \pi^{-1}(\mathfrak{m})$, then by [Sai10], Proposition 4.14, $\mathfrak{q} \supseteq (\Theta_s, \Theta_t + \Theta_u)$ or $(\Theta_t, \Theta_s + \Theta_u)$. If $\mathfrak{q} = (\Theta_s, \Theta_t + \Theta_u)$, then $\nu = \{\Theta_s \leq 0, \Theta_t + \Theta_u \leq 0\}$. If $\mathfrak{q} = (\Theta_t, \Theta_s + \Theta_u)$, then $\nu = \{\Theta_t \leq 0, \Theta_s + \Theta_u \leq 0\}$. They are the two minimal primes mentioned in Example 4.6.

As a corollary of Theorem 4.5, we can describe the fibers $\pi^{-1}(p)$ for every nonzero closed point p in $\text{Spec } S_A$. Let p be in the torus orbit O_τ for some e -dimensional face τ of $\mathbb{R}_{\geq 0}A$, so p corresponds to a semigroup homomorphism $f_p : \mathbb{N}A \rightarrow \mathbb{K}$ with $f_p(a_i) = c_i$, where $c_i = 0$ if and only if $a_i \notin \mathbb{N}(A \cap \tau)$. Then p corresponds to the maximal ideal $\mathfrak{m}_p = (t^{a_1} - c_1, \dots, t^{a_n} - c_n)$ of S_A . The following theorem gives the reduced induced structure of $\pi^{-1}(p)$.

Theorem 4.8. *Under the hypotheses of Theorem 4.5, we have*

$$\frac{\text{gr } D_A}{\sqrt{\mathfrak{m}_p \text{gr } D_A}} \cong S_B \otimes \mathbb{K}[\delta_1, \dots, \delta_e],$$

where S_B is the toric algebra generated by a simplicial scored semigroup

$$\mathbb{N}B \cong \frac{\mathbb{N}A + \mathbb{Z}(A \cap \tau)}{\mathbb{Z}(A \cap \tau)}$$

and $\mathbb{K}[\delta_1, \dots, \delta_e]$ is a polynomial ring in $e (= \dim \tau)$ variables.

Proof. First, notice that t^a , $a \in \mathbb{N}(A \cap \tau)$, acts as a unit on $\frac{\text{gr } D_A}{\sqrt{\mathfrak{m}_p \text{ gr } D_A}}$ because $p \in O_\tau$. So, by abusing the notation

$$\frac{\text{gr } D_A}{\sqrt{\mathfrak{m}_p \text{ gr } D_A}} \cong \frac{\text{gr } D_A[\tau^{-1}]}{\sqrt{\mathfrak{m}_p \text{ gr } D_A[\tau^{-1}]}}$$

where τ^{-1} means we invert t^a for all $a \in \mathbb{N}(A \cap \tau)$. Note also that

$$\text{gr } D_A[\tau^{-1}] \cong \text{gr } D(S_A[\tau^{-1}]) = \text{gr } D(\mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \tau)]).$$

Next, choose a simplicial scored semigroup $\mathbb{N}B$ so that

$$\mathbb{N}A + \mathbb{Z}(A \cap \tau) = \mathbb{N}B \oplus \mathbb{Z}(A \cap \tau).$$

To do this, let's first assume $\mathbb{N}A$ is normal. By an exercise of section 1.2 in [Ful93], $\frac{\mathbb{R}_{\geq 0}A + \mathbb{R}\tau}{\mathbb{R}\tau}$ is a simplicial rational polyhedral cone with facets $\frac{\gamma + \mathbb{R}\tau}{\mathbb{R}\tau}$, where the γ 's are the facets of $\mathbb{R}_{\geq 0}A$ containing τ . So $\frac{\mathbb{N}A + \mathbb{Z}(A \cap \tau)}{\mathbb{Z}(A \cap \tau)}$ is a simplicial normal semigroup in $\mathbb{Z}^d/\mathbb{Z}(A \cap \tau)$. $\mathbb{N}B$ can be obtained by choosing suitable elements in $[\mathbb{N}A + \mathbb{Z}(A \cap \tau)] \setminus \mathbb{Z}(A \cap \tau)$. For the general simplicial scored semigroup $\mathbb{N}A$, we just have to notice that $\widetilde{\mathbb{N}A} \setminus \mathbb{N}A$ is a union of hyperplane sections parallel to some facets of $\mathbb{R}_{\geq 0}A$. So $\mathbb{Z}(A \cap \tau) = \mathbb{Z}(A \cap \tau)$ and $[\mathbb{N}A + \mathbb{Z}(A \cap \tau)] \setminus [\mathbb{N}A + \mathbb{Z}(A \cap \tau)]$ is a union of hyperplane sections parallel to some facets of $\mathbb{R}_{\geq 0}A$ containing τ .

Now,

$$\text{gr } D_A[\tau^{-1}] \cong \text{gr } D(\mathbb{K}[\mathbb{N}B \oplus \mathbb{Z}(A \cap \tau)]) \cong \text{gr } D_B \otimes \text{gr } D_{\mathbb{Z}(A \cap \tau)}.$$

Therefore,

$$\begin{aligned} \frac{\text{gr } D_A}{\sqrt{\mathfrak{m}_p \text{ gr } D_A}} &\cong \frac{\text{gr } D_B \otimes \text{gr } D_{\mathbb{Z}(A \cap \tau)}}{\sqrt{\mathfrak{m}_p \text{ gr } D_B \otimes \text{gr } D_{\mathbb{Z}(A \cap \tau)}}} \\ &\cong \frac{\text{gr } D_B}{\sqrt{\mathfrak{m}_B \text{ gr } D_B}} \otimes \mathbb{K}[\delta_1, \dots, \delta_e] \\ &\cong S_B \otimes \mathbb{K}[\delta_1, \dots, \delta_e] \end{aligned}$$

by Theorem 4.5, where $\delta_1, \dots, \delta_e$ are the standard derivations of $\mathbb{K}[\mathbb{Z}(A \cap \tau)]$. \square

4.2. Characteristic cycles. Let $D := D(R; \mathbb{K})$ as defined in section 2. Let M be a D -module with a filtration $\{M_i\}$ such that $D_i M_j \subseteq M_{i+j}$. The associated graded module $\text{gr } M := \bigoplus M_i/M_{i-1}$ has the natural $\text{gr } D$ -module structure. We call $\{M_i\}$ a good filtration if $\text{gr } M$ is finitely generated over $\text{gr } D$. If M is finitely generated over D by x_1, \dots, x_n , then the filtration $\{\sum_{j=1}^n D_i x_j\}$ is good.

From now on, we assume that $\text{gr } D$ is Noetherian. This is always the case when R is regular.

For a D -module M with a good filtration $\{M_i\}$, define the characteristic variety $\text{Ch}(M)$ of M to be the support of the $\text{gr } D$ -module $\text{gr } M$,

$$\text{i.e. } \text{Ch}(M) = \text{Var}(\text{ann}_{\text{gr } D} \text{gr } M) \subseteq \text{Spec}(\text{gr } D).$$

The characteristic cycle $\text{CC}(M)$ of M is the formal sum of the irreducible components V_i of $\text{Ch}(M)$ counted with multiplicity. More precisely,

$$\text{CC}(M) = \sum m_i V_i,$$

where the multiplicity m_i is the length of the $(\text{gr } D)_{p_i}$ -module $(\text{gr } M)_{p_i}$ and p_i is the prime ideal corresponding to V_i .

$\text{Ch}(M)$ and $\text{CC}(M)$ do not depend on the choice of good filtration. A more detailed discussion about characteristic varieties can be found in [Gin86].

Example 4.9. R is naturally a D -module generated by the identity 1. With the filtration $\{D_i \cdot 1\}$, $\text{gr } R = R$ is the $\text{gr } D$ -module generated by 1. So $R \cong \text{gr } D/\text{ann}_{\text{gr } D}(1)$, and therefore $\text{Ch}(R) = \text{Var}(\text{ann}_{\text{gr } D}(1))$ is abstractly isomorphic to the ambient variety $\text{Spec } R$.

As we mentioned in the introduction, many invariants of $H_I^i(R)$ can be computed via the characteristic cycles when R is a polynomial algebra over \mathbb{K} (see e.g. [ÁM00], [ÁM04]). In this subsection, we compute the characteristic cycles of some local cohomology modules $H_I^i(S_A)$ using our results of finiteness properties in section 3. By the main result in [ST04], $\text{gr } D_A$ is finitely generated over \mathbb{K} if and only if $\mathbb{N}A$ is scored. In particular, $\text{gr } D_A$ is Noetherian when $\mathbb{N}A$ is scored, so it makes sense to talk about characteristic cycles in this case.

Example 4.10. For a 1-dimensional toric algebra $S_A = \mathbb{K}[t^{a_i} \mid i = 1, \dots, n]$, $\text{Ch}(H_I^1(S_A))$ is particularly simple. As in Proposition 4.1,

$$\text{gr } D_A = \mathbb{K} \left[t\xi, t^{|\Omega(w)|} \xi^{|\Omega(-w)|} : |w| \in \{a_1, \dots, a_n\} \cup \text{Hole}(\mathbb{N}A) \right].$$

For any monomial ideal $I \neq 0$ of S_A , $H_I^1(S_A) = \frac{D_A \cdot (1/t)}{S_A}$ by Theorem 3.1. Notice that $t^{a_i} \in \sqrt{\text{ann}_{\text{gr } D_A} \text{gr}(H_I^1(S_A))}$ and that $\xi^{a_i} \notin \sqrt{\text{ann}_{\text{gr } D_A} \text{gr}(H_I^1(S_A))}$. So by Theorem 4.5,

$$\frac{\text{gr } D_A}{\sqrt{\text{ann}_{\text{gr } D_A} \text{gr}(H_I^1(S_A))}} = \mathbb{K}[\delta_i \mid i = 1, \dots, n] \cong S_A,$$

where δ_i is the image of ξ^{a_i} in $\frac{\text{gr } D_A}{\sqrt{\text{ann}_{\text{gr } D_A} \text{gr}(H_I^1(S_A))}}$. Therefore, $\text{Ch}(H_I^1(S_A))$ is abstractly isomorphic to the ambient toric variety $\text{Spec}(S_A)$. Furthermore, in view of the exact sequence

$$0 \rightarrow S_A \rightarrow D_A \cdot (1/t) \rightarrow H_I^1(S_A) \rightarrow 0,$$

we have

$$\text{CC}(S_A[1/f]) = \text{Ch}(S_A) + \text{Ch}(H_I^1(S_A)) \text{ for any monomial } f \in S_A$$

by the additivity of the characteristic cycles on exact sequences. In particular, $\text{Ch}(S_A[1/f])$ has two components, each of which is abstractly isomorphic to $\text{Spec}(S_A)$.

Example 4.11. Consider the toric algebra in Example 3.8. Let $p = \Theta_t$ and $q = \Theta_s$. Then we have

$$\text{gr } D_A = \mathbb{K}[t, ts^2, ts, t^{-1}(2p - q)^2, s(2p - q), t^{-1}s^{-1}(2p - q)q, t^{-1}s^{-2}q^2, s^{-1}q, p, q].$$

Set

$$\begin{aligned} a &= t, b = ts^2, c = ts, d = t^{-1}(2p - q)^2, e = s(2p - q), \\ f &= t^{-1}s^{-1}(2p - q)q, g = t^{-1}s^{-2}q^2, h = s^{-1}q, i = p, j = q, \end{aligned}$$

and consider the surjection $\phi : \mathbb{K}[a, \dots, j] \rightarrow \text{gr } D_A$. Table 1 gives the information about the characteristic cycles. Again, notice that each M is cyclic by Theorem 3.1, so $\text{ann}_{\text{gr } D_A} \text{gr } M$ is easy to compute.

TABLE 1

| M | $J = \phi^{-1}(\text{ann}_{\text{gr } D_A} \text{gr } M)$ | primary decomposition of \sqrt{J} |
|-----------------------|---|---|
| $S_A[1/t]$ | $(f, g, h, i, j) + \text{Ker}\phi$ | $(a, c, f, g, h, i, f, bd - e^2) \cap (d, e, f, g, h, i, j, ab - c^2)$ |
| $H^1_{(t)}(S_A)$ | $(f, g, h, i, j, a) + \text{Ker}\phi$ | $(a, c, f, g, h, i, f, bd - e^2)$ |
| $S_A[1/ts^2]$ | $(d, e, f, i, j) + \text{Ker}\phi$ | $(j, i, h, g, f, e, d, ab - c^2) \cap (j, i, f, c, b, e, d, ag - h^2)$ |
| $H^1_{(ts^2)}(S_A)$ | $(d, e, f, i, j, b) + \text{Ker}\phi$ | $(j, i, f, e, d, c, b, ag - h^2)$ |
| $S_A[1/ts]$ | $(i, j) + \text{Ker}\phi$ | $(j, i, h, c, a, b, e, dg - f^2) \cap (j, i, h, g, f, c, a, bd - e^2)$ $\cap (j, i, h, g, f, e, d, ab - c^2) \cap (j, i, f, c, b, e, d, ag - h^2)$ |
| $H^2_{(t,ts^2)}(S_A)$ | $(a, b, c, e, h, i, j) + \text{Ker}\phi$ | $(a, b, c, e, h, i, j, dg - f^2)$ |

Example 4.12. Consider the toric algebra in Example 4.6 and use the notation there. By Theorem 3.1, we see that

$$J := \phi^{-1}(\text{ann}_{\text{gr } D_A} \text{gr}(H^1_{(s)}(S_A))) = (a, e, f, g, i, j, k, m, n) + \text{Ker}\phi$$

and the primary decomposition of \sqrt{J} is

$$\begin{aligned} \sqrt{J} = & (o, n, m, k, j, i, c, a, f, g, h, e) \cap (n, m, l, k, j, i, d, a, f, g, h, e) \\ & \cap (n, m, k, j, i, d, c, a, f, g, e, bh - lo). \end{aligned}$$

So $\text{CC}(H^1_{(s)}(S_A))$ is a sum of varieties which are not all isomorphic.

To compute $\text{Ch}(H^3_{\mathfrak{m}}(S_A))$, we need

$$J' := \phi^{-1}(\text{ann}_{\text{gr } D_A} \text{gr}(H^3_{\mathfrak{m}}(S_A))) = (a, b, c, d, e, f, g, l, m, n, o) + \text{Ker}\phi$$

and $\sqrt{J'} = (a, b, c, d, e, f, g, l, m, n, o, hi - jk)$.

Inspired by Examples 4.10, 4.11, and 4.12, we have the following.

Theorem 4.13. *For any scored pointed semigroup $\mathbb{N}A$, the characteristic variety $\text{Ch}(H^d_{\mathfrak{m}}(S_A))$ is abstractly isomorphic to the ambient toric variety $\text{Spec}(S_A)$.*

Proof. Since $\mathbb{N}A$ is pointed, by the Ishida complex

$$(4.2) \quad H^d_{\mathfrak{m}}(S_A) = \frac{\mathbb{K}[\mathbb{Z}^d]}{\mathbb{K}[\bigcup_{\sigma:\text{facet}} (\mathbb{N}A + \mathbb{Z}(A \cap \sigma))]}.$$

By Theorem 3.1, $\mathbb{K}[\mathbb{Z}^d] = D_A \cdot (1/t^\alpha)$ for some interior point α , i.e. for some $\alpha \in \mathbb{N}A \setminus [\bigcup_{\sigma:\text{facet}} \mathbb{N}(A \cap \sigma)]$. Consider the expression of $\text{gr } D_A$ in Lemma 4.3. Denote $J := \text{ann}_{\text{gr } D_A} \text{gr}(H^d_{\mathfrak{m}}(S_A))$. Notice that:

- (1) For any $a \in \mathbb{Z}^d \setminus (-\widetilde{\mathbb{N}A})$ and for any facet σ with $F_\sigma(a) > 0$, there exists $n \in \mathbb{N}$ so that $na - \alpha \in \mathbb{N}A + \mathbb{Z}(A \cap \sigma)$. Therefore,

$$t^\alpha \mathbb{K}[\Theta_1, \dots, \Theta_d] \cdot P_a \subseteq \sqrt{J}$$

for all $a \in \mathbb{Z}^d \setminus (-\widetilde{\mathbb{N}A})$.

- (2) For $a = 0 \in \mathbb{Z}^d$, $P_a = 1$ and $\theta_i \cdot t^{-\alpha} = -\alpha_i t^{-\alpha} \in H^d_{\mathfrak{m}}(S_A)$. So by considering the order filtration, $\Theta_i \in J$ for all $i = 1, \dots, d$.
- (3) By exactly the same argument as in item (3) in the proof of Theorem 4.5, we have $t^\alpha P_a \in \sqrt{J}$ for $a \in [-\widetilde{\mathbb{N}A} \setminus (-\mathbb{N}A)]$. Also, for $a \in -\mathbb{N}A \setminus \{0\}$, $t^\alpha P_a$ is a product of some $t^{-\alpha_i} P_{-a_i}$.

- (4) For $a \in -\mathbb{N}A \setminus \{0\}$, $t^a P_a \cdot t^{-\alpha} = P(-\alpha)t^{a-\alpha} \neq 0$ in $\text{gr}(H_{\mathfrak{m}}^d(S_A))$ because α is an interior point and because $a - \alpha \in \mathbb{Z}^d \setminus \bigcup_{\sigma:\text{facet}} \mathbb{K}[\mathbb{N}A + \mathbb{Z}(A \cap \sigma)]$. So $t^a P_a \notin \sqrt{J}$ for $a \in -\mathbb{N}A \setminus \{0\}$.

Therefore,

$$\frac{\text{gr } D_A}{\sqrt{J}} = \mathbb{K} \left[\overline{t^{-a_i} P_{-a_i}} | a_i : \text{columns of } A \right],$$

which is isomorphic to S_A by a similar argument as in the final part of the proof of Theorem 4.5. \square

Remark 4.14. We mention that the description of \sqrt{J} in Theorem 4.13 can be obtained alternatively using Theorem 6.2(2) in [Sai10].

For β , let \mathfrak{m}_β be the ideal $(\theta_1 - \beta_1, \dots, \theta_d - \beta_d)$ of the polynomial ring $\mathbb{K}[\theta_1, \dots, \theta_d]$. Note that $\mathfrak{m}_\beta = \mathfrak{m}_0 + \beta$ in the notation of [Sai10]. Then $L(\mathfrak{m}_\beta) = D_A/I(\mathfrak{m}_\beta)$ is an irreducible D_A -module (see Theorem 4.1.6 in [ST01] and (5.2) in [Sai10]).

Take $\alpha \in \mathbb{Z}^d$ as in the proof of Theorem 4.13. Then we claim that

$$(4.3) \quad H_{\mathfrak{m}_A}^d(S_A) \cong L(\mathfrak{m}_{-\alpha}).$$

Since $\tau(\mathfrak{m}_0 - \alpha) = -\mathbb{R}_{\geq 0}A$, $\sqrt{J} = \sqrt{\text{gr } I(\mathfrak{m}_{-\alpha})} = \mathfrak{P}(\mathfrak{m}_0, -\mathbb{R}_{\geq 0}A)$ by (4.3) and Theorem 6.2(2) in [Sai10].

The claim (4.3) can be proved as follows: By Theorem 3.1, $H_{\mathfrak{m}_A}^d(S_A) = D_A \cdot t^{-\alpha}$. Since $\mathbb{N}A$ is scored, $\mathbb{Z}^d \cap \mathbb{Q}(A \cap \tau) = \mathbb{Z}(A \cap \tau)$ for all faces τ . Hence, for $\beta \in \mathbb{Z}^d$, $\beta \approx -\alpha$ if and only if $\beta \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)$ for some face τ if and only if $\beta \in \mathbb{N}A + \mathbb{Z}(A \cap \sigma)$ for some facet σ . Therefore, we see that $I(\mathfrak{m}_{-\alpha}) \cdot t^{-\alpha} = 0$ by (4.2). Thus there exists a surjective D_A -homomorphism from $L(\mathfrak{m}_{-\alpha})$ to $H_{\mathfrak{m}_A}^d(S_A)$. Since $L(\mathfrak{m}_{-\alpha})$ is irreducible, it is an isomorphism.

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