

A COUNTEREXAMPLE FOR SUBADDITIVITY OF MULTIPLIER IDEALS ON TORIC VARIETIES

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We construct a 3-dimensional complete intersection toric variety on which the subadditivity formula doesn't hold, answering negatively a question by Takagi and Watanabe. A combinatorial proof of the subadditivity formula on 2-dimensional normal toric varieties is also provided.

Key Words: Multiplier ideals; Subadditivity formula; Toric varieties.

2010 Mathematics Subject Classification: 14F18; 14M25.

1. INTRODUCTION

Demailly et al. [2] proved the subadditivity theorem for multiplier ideals on smooth complex varieties, which states

$$\mathcal{F}(\alpha\mathfrak{b}) \subseteq \mathcal{F}(\alpha)\mathcal{F}(\mathfrak{b}).$$

This theorem is responsible for several applications of multiplier ideals in commutative algebra, in particular to symbolic powers [3] and Abhyankar valuations [4].

In a later article, Takagi and Watanabe [9] investigated the extent to which the subadditivity theorem remains true on singular varieties. They showed that on \mathbb{Q} -Gorenstein normal surfaces, the subadditivity formula holds if and only if the variety is log terminal [9, Theorem 2.2]. Furthermore, they gave an example of a \mathbb{Q} -Gorenstein normal toric threefold on which the formula is not satisfied [9, Example 3.2]. This led Takagi and Watanabe to ask the following question.

Question 1.1. Let R be a Gorenstein toric ring and α, \mathfrak{b} be monomial ideals of R . Is it true that

$$\mathcal{F}(\alpha\mathfrak{b}) \subseteq \mathcal{F}(\alpha)\mathcal{F}(\mathfrak{b})?$$

Received August 17, 2010; Revised December 27, 2010. Communicated by I. Swanson.
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The purpose of this article is to provide a counterexample to Question 1.1. We will also give, in Section 4, a combinatorial proof of the subadditivity formula on any 2-dimensional normal toric rings. The standard notation and facts in [5] will be used freely in the presentation.

2. MULTIPLIER IDEALS ON TORIC VARIETIES

Let \mathbb{K} be a field and $R = \mathbb{K}[M \cap \sigma^\vee]$ be the coordinate ring of an affine normal Gorenstein toric variety. Denote $X = \text{Spec}(R)$. In this case, the canonical divisor K_X of X is Cartier, so there exists a $u_0 \in M \cap \sigma^\vee$ such that $(u_0, n_i) = 1$ where the n_i 's are the primitive generators of σ . For any monomial ideal α of R , denote $\text{Newt}(\alpha)$ the Newton polyhedron of α and $\text{relint Newt}(\alpha)$ the relative interior of $\text{Newt}(\alpha)$. The multiplier ideal $\mathcal{F}(\alpha)$ of α in R admits a combinatorial description.

Proposition 2.1.

$$\mathcal{F}(\alpha) = \langle \underline{x}^w \in R \mid w + u_0 \in \text{relint Newt}(\alpha) \rangle. \tag{2.1}$$

This is a result by Hara and Yoshida [7, Theorem 4.8] which is generalized by Blickle [1] to arbitrary normal toric varieties.

3. THE EXAMPLE

Consider the 3-dimensional normal semigroup ring $R = \mathbb{K}[x^2y, xy, xy^2, z]$, \mathbb{K} a field. Notice that R is a complete intersection, and hence Gorenstein. Note also that

$$u_0 = (1, 1, 1).$$

Consider the following two ideals of R :

$$\alpha = \langle x^2y^4, x^{10}y^6z^2 \rangle,$$

$$\mathfrak{b} = \langle x^{12}y^7, x^{10}y^6z^2 \rangle.$$

Then $\alpha\mathfrak{b} = \langle x^{14}y^{11}, x^{12}y^{10}z^2, x^{22}y^{13}z^2, x^{20}y^{12}z^4 \rangle$. Denote

$$w_1 = (14, 11, 0),$$

$$w_2 = (12, 10, 2),$$

$$w_3 = (22, 13, 2),$$

$$w_4 = (20, 12, 4).$$

Observe that the lattice point

$$v = (18, 12, 2) \in \text{relint Newt}(\alpha\mathfrak{b}).$$

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To see this, consider the four points

$$\begin{aligned}v_1 &= w_1 = (14, 11, 0), \\v_2 &= w_1 + (4, 2, 0) = (18, 13, 0), \\v_3 &= w_1 + (2, 1, 4) = (16, 12, 4), \\v_4 &= \frac{1}{2}(w_3 + w_4) = \left(21, \frac{25}{2}, 3\right).\end{aligned}$$

They are in $\text{Newt}(\alpha\mathfrak{b})$ and do not lie on a plane, namely, they are affinely independent. Since

$$v = \frac{5}{16}v_1 + \frac{1}{16}v_2 + \frac{1}{8}v_3 + \frac{1}{2}v_4,$$

it is in $\text{relint Newt}(\alpha\mathfrak{b})$.

Now, since $-u_0 + v = (17, 11, 1)$, by (2.1)

$$x^{17}y^{11}z \in \mathcal{F}(\alpha\mathfrak{b}).$$

We claim that

$$x^{17}y^{11}z \notin \mathcal{F}(\alpha)\mathcal{F}(\mathfrak{b}).$$

An element in $\mathcal{F}(\alpha)\mathcal{F}(\mathfrak{b})$ is a finite sum of monomials of the form $c \cdot \underline{x}^\alpha \underline{x}^\beta$ where $c \in k$, $\alpha, \beta \in M \cap \sigma^\vee$, $\alpha + u_0 \in \text{relint Newt}(\alpha)$, and $\beta + u_0 \in \text{relint Newt}(\mathfrak{b})$. If $x^{17}y^{11}z \in \mathcal{F}(\alpha)\mathcal{F}(\mathfrak{b})$, then

$$-u_0 + v = \alpha + \beta$$

for some α, β as above. This means $v = (18, 12, 2)$ can be written as a sum of a lattice point $(\alpha + u_0)$ in $\text{relint Newt}(\alpha)$ and a lattice point β in $-u_0 + \text{relint Newt}(\mathfrak{b})$. We check that this is not possible.

Suppose α and β are lattice points satisfying $\alpha + u_0 + \beta = v = (18, 12, 2)$. Write $\alpha' = \alpha + u_0 = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$, so

$$(a_1 + b_1, a_2 + b_2, a_3 + b_3) = v = (18, 12, 2).$$

We will show that in each case either $\alpha' \notin \text{relint Newt}(\mathfrak{b})$ or $\beta + u_0 \notin \text{relint Newt}(\mathfrak{b})$. First, note that the Newton polyhedron $\text{Newt}(\alpha)$ is the intersection of halfspaces determined by the following five hyperplanes: $2x - y = 0$, $-x + 4y = 14$, $-x + 2y = 2$, $-x + 2y + 2z = 6$, $z = 0$. So we have

$$\begin{aligned}\text{relint Newt}(\alpha) &= \{(x, y, z) \in M \mid 2x - y > 0, -x + 4y > 14, -x + 2y > 2, \\ &\quad -x + 2y + 2z > 6, z > 0\}.\end{aligned}\tag{3.1}$$

Also, $\text{Newt}(b)$ is the intersection of the halfspace determined by the following four hyperplanes: $2x - y = 14$, $-x + 2y = 2$, $4x - 2y + 3z = 34$, $z = 0$. We have

$$\begin{aligned} \text{relint Newt}(\alpha) = \{ & (x, y, z) \in M \mid 2x - y > 14, -x + 2y > 2, \\ & 4x - 2y + 3z > 34, z > 0\}. \end{aligned} \tag{3.2}$$

We consider the following cases:

Case I: If $a_1 \geq 7$, then $b_2 \leq 5$ and $\beta + u_0 \notin \text{relint Newt}(b)$. To see this, suppose $\beta + u_0 = (b_1 + 1, b_2 + 1, b_3 + 1) \in \text{relint Newt}(b)$. By (3.2), $2(b_1 + 1) - (b_2 + 1) > 14$ and $-(b_1 + 1) + 2(b_2 + 1) > 2$. So $4(b_2 + 1) - 4 > 2(b_1 + 1) > 14 + (b_2 + 1)$ and hence $b_2 > 5$, which is a contradiction.

Case II. If $a_2 \leq 4$, then $\alpha' \notin \text{relint Newt}(\alpha)$. Indeed, suppose $\alpha' = (a_1, a_2, a_3) \in \text{relint Newt}(\alpha)$. By (3.1), $2a_1 - a_2 > 0$ and $-a_1 + 4a_2 > 14$. So $8a_2 - 28 > 2a_1 > a_2$ and hence $a_2 > 4$.

Case III. Suppose $a_2 = 5$ and $b_2 = 7$.

- a) If $a_1 \geq 6$, then $\alpha' \notin \text{relint Newt}(\alpha)$. Indeed, suppose $\alpha' = (a_1, a_2, a_3) \in \text{relint Newt}(\alpha)$. By (3.1), $-a_1 + 4a_2 > 14$ and hence $a_1 < 4a_2 - 14 = 6$.
- b) If $a_1 \leq 5$, then $b_1 \geq 13$. This implies $\beta + u_0 \notin \text{relint Newt}(b)$. Indeed, suppose $\beta + u_0 = (b_1 + 1, b_2 + 1, b_3 + 1) \in \text{relint Newt}(b)$. By (3.2), $-(b_1 + 1) + 2(b_2 + 1) > 2$ and hence $b_1 < 2(b_2 + 1) - 3 = 13$.

Case IV: Suppose $a_2 = b_2 - 6$.

- a) If $b_1 \neq 10$, then $\beta + u_0 \notin \text{relint Newt}(b)$. To see this, suppose $\beta + u_0 = (b_1 + 1, b_2 + 1, b_3 + 1) \in \text{relint Newt}(b)$. By (3.2) again, $2(b_1 + 1) - (b_2 + 1) > 14$ and $-(b_1 + 1) + 2(b_2 + 1) > 2$. This forces $b_1 = 10$.
- b) If $b_1 = 10$, then $\alpha' = (a_1, a_2, a_3) = (8.6, a_3)$ and $\beta = (b_1, b_2, b_3) = (10, 6, b_3)$.
 - i) If $a_3 \leq 0$, then $\alpha' \notin \text{relint Newt}(\alpha)$ by (3.1).
 - ii) If $a_3 > 2$, then $b_3 < 0$. In this case, $\beta + u_0 \notin \text{relint Newt}(b)$ by (3.2).
 - iii) If $\alpha' = (a_1, a_2, a_3) = (8, 6, 1)$, then $-a_1 + 2a_2 + 2a_3 = 6$. So $\alpha' \notin \text{relint Newt}(\alpha)$ by (3.1).
 - iv) If $\alpha' = (a_1, a_2, a_3) = (8, 6, 2)$, then $\beta = (b_1, b_2, b_3) = (10, 6, 0)$. So $4(b_1 + 1) - 2(b_2 + 1) + 3(b_3 + 1) = 33 < 34$. Hence $\beta + u_0 \notin \text{relint Newt}(b)$ by (3.2).

Remark 3.1. We briefly explain the idea behind the example. Recall that the integral closure \bar{I} of a monomial ideal I in a normal toric ring R is determined by $\text{Newt}(I)$ (see, for example, [8]):

$$\bar{I} = \langle \underline{x}^w \in R \mid w \in \text{Newt}(I) \rangle.$$

So Question 1.1 is closely related to the containment $\bar{I} \cdot \bar{J} \subseteq \overline{IJ}$ for monomial ideals of R . Huneke and Swanson provide a trick to construct examples where the strict containment $\bar{I} \cdot \bar{J} \subsetneq \overline{IJ}$ occur (see [6, Example 1.4.9] and the remark after it). We repeat their construction here:

Choose a ring R' and a pair of ideal I', J' in R' such that

$$\overline{I'} + \overline{J'} \subsetneq \overline{I' + J'}.$$

Pick an element

$$r \in \overline{I' + J'} \setminus (\overline{I'} + \overline{J'}).$$

Set $R = R'[Z]$ for some variable Z over R' and set

$$I = \overline{I'}R + ZR, J = \overline{J'}R + ZR.$$

Then I and J are integrally closed and

$$rZ \in \overline{IJ} \setminus \overline{I} \cdot \overline{J}.$$

This kind of construction doesn't always guarantee a counterexample to Question 1.1. However, a suitable choice of $r, Z, R', I',$ and J' will do. In our example, take

$$R' = \mathbb{K}[x^2y, xy, xy^2],$$

$$r = x^8y^6,$$

$$I' = \langle x^2y^4 \rangle,$$

$$J' = \langle x^{12}y^7 \rangle,$$

$$Z = x^{10}y^6z^2.$$

Then $rZ = x^{18}y^{12}z^2$ is exactly the crucial point we considered in the example.

4. TWO-DIMENSIONAL CASE

Let $R = \mathbb{K}[M \cap \sigma^\vee]$, \mathbb{K} a field, be a 2-dimensional normal toric ring and denote $X = \text{Spec}(R)$. Then there exists a primitive lattice point $w_0 \in M \cap \sigma^\vee$ such that $(w_0, n_i) = r \in \mathbb{Z}_{\geq 0}$ where the n_i 's are the primitive generators of σ . So the canonical divisor K_X of X is \mathbb{Q} -Cartier and R is \mathbb{Q} -Gorenstein.

Set $u_0 = w_0/r$. By Theorem 4.8 in [7], for any monomial ideal α in R

$$\mathcal{F}(\alpha) = \langle \underline{x}^w \in R \mid w + u_0 \in \text{relint Newt}(\alpha) \rangle. \quad (4.1)$$

The following theorem establishes the subadditivity formula on two-dimensional normal toric rings.

Theorem 4.1. *For any pair of monomial ideal α, \mathfrak{b} in R ,*

$$\mathcal{F}(\alpha\mathfrak{b}) \subseteq \mathcal{F}(\alpha)\mathcal{F}(\mathfrak{b}).$$

Proof. Write $\alpha = \langle \underline{x}^a \mid a \in A \rangle$ and $\mathfrak{b} = \langle \underline{x}^b \mid b \in B \rangle$ for some finite sets A and B in $M \cap \sigma^\vee$. We assume that $\{x^a \mid a \in A\}$ and $\{x^b \mid b \in B\}$ are the sets of monomial minimal generators of α and \mathfrak{b} , respectively. Then $\alpha\mathfrak{b} = \langle \underline{x}^{a+b} \mid a \in A \text{ and } b \in B \rangle$. Let $\alpha_1, \dots, \alpha_k$ be the vertices of the Newton polyhedron $\text{Newt}(\alpha\mathfrak{b})$ such that

$$\alpha_1 + \rho_1, \text{conv}\{\alpha_1, \alpha_2\}, \dots, \text{conv}\{\alpha_{k-1}, \alpha_k\}, \quad \text{and} \quad \alpha_k + \rho_2$$

form the boundary of $\text{Newt}(\alpha\mathfrak{b})$, where ρ_1, ρ_2 are the two rays of σ^\vee . Then

$$\text{Newt}(\alpha\mathfrak{b}) = \bigcup_{i=1}^{k-1} (\text{conv}\{\alpha_i, \alpha_{i+1}\} + \sigma^\vee).$$

Note also that the α_i 's are of the form $a_i + b_i$ for some $a_i \in A$ and $b_i \in B$. Suppose that for some $i \in \{1, \dots, k-1\}$, we have $a_i \neq a_{i+1}$ and $b_i \neq b_{i+1}$. Then $a_i + b_{i+1} = a_{i+1} + b_i$, lie on boundary segment $\text{conv}\{\alpha_i, \alpha_{i+1}\}$, since otherwise they lie on different sides of $\text{conv}\{\alpha_i, \alpha_{i+1}\}$ which is a contradiction. For any such i , we insert the point $a_i + b_{i+1}$ to the sequence $\alpha_1, \dots, \alpha_k$. So we obtain a sequence, say $\beta_1 = a'_1 + b'_1, \dots, \beta_s = a'_s + b'_s$, such that, for each $i \in \{1, \dots, s-1\}$, either $a'_i = a'_{i+1}$ or $b'_i = b'_{i+1}$, and that

$$\text{Newt}(\alpha\mathfrak{b}) = \bigcup_{i=1}^{s-1} (\text{conv}\{\beta_i, \beta_{i+1}\} + \sigma^\vee).$$

Now, observe that

$$\text{relint Newt}(\alpha\mathfrak{b}) \subseteq \bigcup_{i=1}^{s-1} (\text{relint } \Delta_i),$$

where $\Delta_i = \text{conv}\{\beta_i, \beta_{i+1}\} + \sigma^\vee$. If $\underline{x}^p \in \mathcal{F}(\alpha\mathfrak{b})$, then by (4.1) $p + u_0 \in \text{relint Newt}(\alpha\mathfrak{b})$ and hence in $\text{relint } \Delta_{i_0}$ for some i_0 . Without loss of generality, we may assume $a'_{i_0} = a'_{i_0+1}$. So

$$p + u_0 \in \text{relint } \Delta_{i_0} = a'_{i_0} + [\text{relint}(\text{conv}\{b'_{i_0}, b'_{i_0+1}\} + \sigma^\vee)] \subseteq a'_{i_0} + \text{relint Newt}(\mathfrak{b}).$$

Therefore, $p \in a'_{i_0} + [-u_0 + \text{relint Newt}(\mathfrak{b})]$. Since $a'_{i_0} + u_0 \in \text{relint Newt}(\alpha)$, by (4.1) we conclude that $\underline{x}^p \in \mathcal{F}(\alpha)\mathcal{F}(\mathfrak{b})$, as desired. \square

Remark 4.2. As one can see in the proof of Theorem 4.1, the choice of β_i 's is essential. For any $\underline{x}^p \in \mathcal{F}(\alpha\mathfrak{b})$ we are able to choose $a \in \text{Newt}(\alpha)$ such that x^a is in the set of monomial minimal generators of α and that $p + u_0 \in \text{relint Newt}(\mathfrak{b})$. This cannot be extended to the higher dimensional case. From the example in Section 3, $x^{17}y^{11}z \in \mathcal{F}(\alpha\mathfrak{b})$ and $u_0 = (1, 1, 1)$. $\text{Newt}(\alpha)$ is minimally generated by x^2y^4 and $x^{10}y^6z^2$. But $(16, 8, 2) = (18, 12, 2) - (2, 4, 0)$ and $(8, 6, 0) = (18, 12, 2) - (10, 6, 2)$ are not in $\text{relint Newt}(\mathfrak{b})$ by (3.2). Similarly, $\text{Newt}(\mathfrak{b})$ is minimally generated by $x^{12}y^7$ and $x^{10}y^6z^2$. But $(6, 5, 2) = (18, 12, 2) - (12, 7, 0)$ and $(8, 6, 0) = (18, 12, 2) - (10, 6, 2)$ are not in $\text{relint Newt}(\alpha)$ by (3.1).

ACKNOWLEDGMENTS

The author was partially supported by NSF under grant DMS 0555319 and DMS 0901123. The author would like to thank his advisor, Uli Walther, for his encouragement during the preparation of this work. He is also grateful to the referee for the careful reading and useful suggestions.

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