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A remark on bigness of the tangent bundle of a smooth projective variety and D-simplicity of its section rings

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We point out a connection between bigness of the tangent bundle of a smooth projective variety X over $\mathbb{C}$ and simplicity of the section rings of X as modules over their rings of differential operators. As a consequence, we see that the tangent bundle of a smooth projective toric variety or a (partial) flag variety is big. Some other applications and related questions are discussed.

Keywords: Big vector bundle; tangent bundle; D-simple ring.

Mathematics Subject Classification: 13N10, 14J60

1. Introduction

Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and T_X be the tangent bundle of X. A well-known theorem of Mori [17] says that if T_X is ample, then X is isomorphic to the projective space $\mathbb{P}^n$. In [3], Campana and Peternell initiated a program to characterize projective manifolds whose tangent bundles are numerically effective (nef). In view of the theory of positivity of vector bundles [13, 14], it is natural to consider the following.

Problem 1.1. *Classify projective manifolds whose tangent bundles are big.*

Recall that a line bundle L on X is big if and only if the map

$$\phi_m : X \dashrightarrow \mathbb{P}H^0(X, L^{\otimes m})$$

defined by $L^{\otimes m}$ is birational onto its image for some $m > 0$ [14, Definition 2.2.1]. A vector bundle E on X is big if the line bundle $\mathcal{O}(1)$ on the projective bundle

$\mathbb{P}(E) := \mathrm{Proj}(\mathrm{Sym}\, E)$ is big. Except for the case where E is nef (see the discussion in Sec. 4), it is in general not clear how to determine the bigness of a vector bundle. Here, we propose the following criterion for bigness of the tangent bundle T_X.

Theorem 1.2. *If X admits an ample line bundle whose section ring S is simple as a module over its ring of $\mathbb{C}$-linear differential operators D_S, then T_X is a big vector bundle.*

In fact, it is shown that if T_X is not big, then the section ring S associated to any ample line bundle on X has no differential operators of negative weight. In particular, the maximal graded ideal of S is a D_S-submodule (S is not D_S-simple) and D_S itself is not a simple ring.

When $\dim X = 1$, Theorem 1.2 was proved by Levasseur [15] following the results in [1, 12]. The essential ideas due to Bernšteĭn, Gel′fand, Gel′fand [1] show that there exist no differential operators of negative weight on

$$S = \mathbb{C}[x, y, z]/\langle x^3 + y^3 + z^3\rangle$$

by using the geometric properties of the cubic curve $\mathrm{Proj}\, S$. These ideas generalize readily to the higher-dimensional cases (see Sec. 2).

As the first consequence of Theorem 1.2, we have the following corollary whose proof can be found in Sec. 3.

Corollary 1.3. *The tangent bundle on a smooth projective toric variety or on a* (*partial*) *flag variety is big.*

Another application of Theorem 1.2 is that if T_X is not big and if S is a quasi-Gorenstein section ring on X, then the multiplier ideal $\mathcal{J}(S)$ is a D_S-module (see Remark 3.4). This can be regarded as a characteristic 0 analog of Theorem 2.2(1) in [19]: every test ideal is a D-module.

In the case where T_X is nef, the bigness of T_X is equivalent to the positivity of the Segre class $s_n(T_X^*)$ [8, Sec. 3.1]. Using this and the results in [3], we show in Proposition 4.1 that if T_X is nef and big then X is Fano (i.e. $-K_X$ is ample). This provides higher-dimensional examples of smooth projective varieties whose section rings are not D-simple (e.g. abelian varieties or other non-Fano manifolds with nef T_X), generalizing the classical one-dimensional examples in [1, 15]. On the other hand, Proposition 4.1 gives a partial solution to Problem 1.1 when T_X is nef. In fact, following the discussions in Sec. 5 we expect that the tangent bundle T_X on a Fano manifold X is always big. When $\dim X \leq 3$ and T_X is nef, this is verified by computing the Segre class $s_n(T_X^*)$ using the classification in [3]. See the paragraph after Question 4.5 for more details.

To end the introduction, we mention that inspired by the proof of Proposition 4.1, one can prove a conjecture in [3] that states: if T_X is nef and $\chi(\mathcal{O}_X) = 1$ then X is Fano (see Theorem 4.3).

2. The Proof of Theorem 1.2

Except for Proposition 2.1, most parts of the following proof of Theorem 1.2 come from the arguments in [1, 12, 15]. We reproduce them here for the convenience of the reader.

Fix an ample line bundle L on X and set $S_i := H^0(X, L^i)$. Consider the associated section ring

$$S = S_L := \bigoplus_{i \in \mathbb{Z}_{\geq 0}} S_i.$$

Since X is smooth, it is well-known that S is a $\mathbb{N}$-graded normal domain that is finitely generated over $\mathbb{C} = S_0$. The maximal graded ideal of S will be denoted by

$$\mathfrak{m} := \bigoplus_{i \in \mathbb{N}} S_i.$$

Choose a representation T/J of S, where $T = \mathbb{C}[x_1, \ldots, x_s]$ is a polynomial ring whose variable x_i has weight d_i and J is a homogeneous prime ideal. Thanks to the ampleness of L, the graded ring S satisfies the condition ($\sharp$) in [12] which states that there exists l_0 such that for every $l \geq l_0$, $S^{(l)} = \bigoplus_{i \in \mathbb{N}} S_{il}$ is generated by $S_l = [S^{(l)}]_1$ over $S_0 = \mathbb{C}$.

Let D_S be the ring of $\mathbb{C}$-linear differential operators on S in the sense of Grothendieck. By definition, the ring D_S has an increasing filtration $\{D_m\}_{m \geq 0}$ by the order of the differential operators, where $D_0 = S$ and for $m > 0$

$$D_m = \{\delta \in \operatorname{Hom}_{\mathbb{C}}(S, S) \,|\, [\delta, s] \in D_{m-1} \text{ for all } s \in S\}.$$

Following [1] or [18], we decompose D_S as

$$D_S = S \bigoplus \bigcup_{m \geq 1} \operatorname{Der}^m(S),$$

where

$$\operatorname{Der}^m(S) := \{\delta \in D_S \,|\, \operatorname{ord} \delta \leq m \text{ and } \delta(1) = 0\}$$

is the set of all derivations of order $\leq m$. An element $\delta \in \operatorname{Der}^m(S)$ is called homogeneous of weight l ($l \in \mathbb{Z}$) if $\delta(S_i) \subseteq S_{i+l}$ for all i. In particular, the Euler derivation

$$I = \sum_{i=1}^{s} d_i x_i \frac{\partial}{\partial x_i} \in \operatorname{Der}^1(S)$$

is homogeneous of weight 0. Moreover, $\operatorname{Der}^m(S)$ can be decomposed as

$$\operatorname{Der}^m(S) = \bigoplus_{l \in \mathbb{Z}} \operatorname{Der}^m_l(S),$$

where $\operatorname{Der}^m_l(S)$ is the space of homogeneous derivations of S with weight l and order $\leq m$. Notice that if

$$\operatorname{Der}^m_l(S) = 0 \quad \text{for all } m \geq 0 \text{ and } l < 0, \tag{2.1}$$

then the maximal ideal $\mathfrak{m}$ is a D_S-submodule of S; in particular, the ring S is not D_S-simple. Our goal is to show that non-bigness of T_X implies the condition (2.1).

Let $\hat{X} = \mathrm{Spec}(S)$ be the cone over X and $U := \hat{X}\backslash\{\mathfrak{m}\}$ be the punctured spectrum. We have the natural projection

$$\pi : U \to X = \mathrm{Proj}\, S.$$

Let $\mathcal{D}er^m$ be the sheaf of derivations on U of order $\leq m$ and let $\mathcal{D}er^m_l$ be the sheaf of derivations δ on U of order $\leq m$ that satisfy the condition $[I,\delta] = I\delta - \delta I = l\delta$ and denote $\Delta^m_l := \pi_*\mathcal{D}er^m_l$. Since $\mathrm{Der}^m(S)$ is the dual of the mth-order differential module $\Omega^m(S) = \mathcal{I}/\mathcal{I}^{m+1}$ where $\mathcal{I}$ is the diagonal ideal of $S \otimes_{\mathbb{C}} S$, it is reflexive. Moreover, since S is normal, standard facts about reflexive sheaves [10, Proposition 1.6] implies that $\Gamma(U, \mathcal{D}er^m) = \mathrm{Der}^m(S)$. In particular, by comparing the weight of derivations we have

$$\Gamma(X, \Delta^m_l) = \Gamma(U, \mathcal{D}er^m_l) = \mathrm{Der}^m_l(S).$$

For $|l| \geq l_0$ in condition ($\sharp$), one can check locally that $\Delta^m_l \cong \Delta^m_0 \otimes L^l$ $(m \geq 0)$ where this isomorphism is compatible with the natural embedding $\Delta^m_l \hookrightarrow \Delta^n_l$ $(m < n)$ [12, Lemma 6]. In particular, we have the following exact sequence

$$0 \to \Delta^{m-1}_l \to \Delta^m_l \to \sigma_m \otimes L^l \to 0 \quad (m \geq 2, |l| \geq l_0). \tag{2.2}$$

Here $\sigma_m := \Delta^m_0/\Delta^{m-1}_0$ $(m \geq 2)$ and we set $\sigma_1 = \Delta^1_0$.

By [12, Lemma 6(3)], we have the following short exact sequence that comes from the Euler derivation

$$0 \to \mathcal{O}_X \xrightarrow{\cdot I} \sigma_1 \to T_X \to 0. \tag{2.3}$$

Taking symmetric power and using the facts that $\sigma_m = \mathrm{Sym}^m \sigma_1$ [12, Lemma 6(3)] and T_X are locally free, we have the exact sequence [12, Lemma 7]

$$0 \to \sigma_{m-1} \to \sigma_m \to \mathrm{Sym}^m T_X \to 0 \quad (m \geq 2). \tag{2.4}$$

On the other hand, we need the following.

Proposition 2.1. *If T_X is not big, then*

$$H^0(X, \mathrm{Sym}^m T_X \otimes L^l) = 0 \quad \textit{for } m \geq 1 \textit{ and } l < 0.$$

Proof. This can be achieved by a similar argument as in [14, Example 6.1.23]: Suppose $H^0(X, \mathrm{Sym}^m T_X \otimes A^{-1}) \neq 0$ for some m and some ample line bundle A on X. Since $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}(T_X)}(1)$ is π-ample, $\mathcal{O}(1) \otimes \pi^*(A^{\otimes k})$ is ample for k sufficiently large. Denote $\xi = c_1(\mathcal{O}(1))$ and $\alpha = c_1(A)$. By the projection formula, $\pi_*(\mathcal{O}(m) \otimes \pi^* A^{-1}) = \mathrm{Sym}^m T_X \otimes A^{-1}$. So $H^0(\mathbb{P}(T_X), \mathcal{O}(m) \otimes \pi^* A^{-1}) \neq 0$ and hence $m\xi - \pi^*\alpha$ is an effective class. For k sufficiently large,

$$(km+1)\xi = [km\xi - \pi^*(k\alpha)] + [\xi + \pi^*(k\alpha)]$$

is a sum of an effective class and an ample class. Therefore, ξ is big. □

From now, assume that T_X is not big. By Proposition 2.1, we have

$$H^0(X, T_X \otimes L^l) = 0 \quad \text{for } l < 0.$$

The global evaluation of (2.3) tensored with L^l together with the Kodaira vanishing theorem imply that

$$\mathrm{Der}_l^1(S) = H^0(X, \Delta_l^1) = 0 \quad (l < 0).$$

Moreover, the global evaluation of (2.4) tensored with L^l together with the vanishing

$$H^0(X, \mathrm{Sym}^m T_X \otimes L^l) = 0 \quad (m \geq 2, l < 0)$$

imply that

$$H^0(X, \sigma_m \otimes L^l) = H^0(X, \sigma_1 \otimes L^l) = H^0(X, \Delta_l^1) = 0 \quad (m \geq 1, l < 0).$$

Therefore, it follows from the global evaluation of $(2.2) \otimes L^l$ that

$$\mathrm{Der}_l^m(S) = H^0(X, \Delta_l^m) = H^0(X, \Delta_l^1) = 0 \quad (m \geq 1, l \leq -l_0).$$

Now, for any $\delta \in \mathrm{Der}_l^m(S) (m \geq 1, l < 0)$, we have $\delta^k \in \mathrm{Der}_{kl}^{km}(S) = 0$ for k sufficiently large. This forces $\delta = 0$, and the proof of Theorem 1.2 is finished.

3. Some Applications

The following fact ([19, Proposition 3.1] or [16, Proposition 3.5]) says that D-simplicity is preserved when considering pure subrings.

Proposition 3.1. *Let S and T be arbitrary commutative algebra over a commutative ring A. Suppose that $S \hookrightarrow T$ and that this inclusion splits as a map of S-modules. If T is a simple $D_A(T)$-module, then S is a simple $D_A(S)$-module.*

In particular, if S admits a split embedding into a polynomial algebra $\mathbb{C}[x_1, \ldots, x_t]$, then S is D_S-simple.

We are now ready to prove the main corollary in the introduction which states: the tangent bundle on a smooth projective toric variety or on a (partial) flag variety is always big.

Proof of Corollary 1.3. For the toric case, this follows from the facts that every section ring of a smooth projective toric variety is a normal semigroup ring [6, Sec. 3.4] and that every positive normal semigroup ring admits a split embedding into a polynomial algebra [2, 6.1.10]. Alternatively, one can check directly that every normal semigroup ring S is D_S-simple (see e.g. [11, Theorem 3.7]).

For the flag case, note that the homogeneous coordinate ring of a (partial) flag variety under the Plücker embedding is isomorphic to the ring of invariants R^G of a polynomial algebra R under the action of certain linearly reductive group G [7, Sec. 9.2]. The existence of a Reynolds operator $\rho : R \to R^G$ [2, Sec. 6.5] guarantees that the embedding $R^G \to R$ is split. □

Remark 3.2. The flag case of Corollary 1.3 recovers the bigness of $T_{\mathbb{P}(T_{\mathbb{P}^2})}$ in [5, Example 2].

Remark 3.3. It is pointed out by Mustaţă that the toric case of Corollary 1.3 can also be recovered by using the Euler exact sequence. On the other hand, the pseudoeffective cone of a projectivized toric vector bundle $\mathbb{P}(\mathcal{F})$ on a toric variety X is described using Klyachko filtration [9]. In the case where $\mathcal{F}$ is the tangent bundle T_X, the pseudoeffective cone of $\mathbb{P}(T_X)$ is generated by the classes $\mathcal{D}_H$, for hypersurfaces H in $\mathbb{P}_F$ (the π-fiber over 1_T), and the classes D_i (the preimage under π of the torus invariant prime divisors on X). As Payne mentioned to us, the results in [9] might be helpful to re-establish the bigness of tangent bundles on smooth projective toric varieties.

We mention another consequence of Theorem 1.2.

Remark 3.4. For a normal standard graded ring S that is quasi-Gorenstein and has an isolated singularity, the multiplier ideals $\mathcal{J}(S)$ of S is a power of the maximal ideal $\mathfrak{m}$ [21, Example 3.7]. Therefore, if the section ring S in Theorem 1.2 is quasi-Gorenstein, then $\mathcal{J}(S)$ is a D_S-submodule of S. This can be regarded as a characteristic 0 analog of [19, Theorem 2.2(1)]: every test ideal is a D-module.

4. The Case where T_X is nef

Recall that a vector bundle E of rank $r+1$ on X is numerically effective (nef) if the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$ is nef, i.e. $c_1(\mathcal{O}(1)) \cdot C \geq 0$ for all curves C in $\mathbb{P}(E)$. In this case, the vector bundle E is big if and only if the volume of $\mathcal{O}_{\mathbb{P}(E)}(1)$

$$\int_{\mathbb{P}(E)} c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{n+r} = \int_X s_n(E^*) > 0,$$

where $s_n(E^*)$ is the nth Segre class of the dual of E [13, Sec. 2.2.C], [8, Sec. 3.1]. The reader is cautioned that in [8], $\mathbb{P}(E)$ denotes the projective bundle of lines in E.

The study of projective manifolds with nef tangent bundle was initiated in [3]. In general, if T_X is nef, then the Chern classes

$$c_i := c_i(X) = c_i(T_X) \geq 0.$$

In this case, we have $(-K_X)^n = c_1^n \geq 0$ and the inequality is strict if and only if X is Fano (i.e. $-K_X$ is ample) [4, Theorem 1.2]. Moreover, the Schur polynomial [8, Example 12.1.7]

$$\Delta_\lambda = \det(c_{\lambda_i+j-i})_{1\leq i,j\leq n} \geq 0$$

for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of n with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Note that by [8, Lemma A.9.2]

$$s_n(T_X^*) = \Delta_{(1^n)}, \quad \text{where } (1^n) = (1, 1, \ldots, 1),$$

so T_X is big if and only if $\Delta_{(1^n)} > 0$. On the other hand, for a partition $\mu = (\mu_1, \ldots, \mu_n)$ of n with $n \geq \mu_1 \geq \cdots \geq \mu_n \geq 0$, denote

$$c_\mu = c_{\mu_1} \cdot c_{\mu_2} \cdot \cdots \cdot c_{\mu_n}.$$

It follows from the Pieri's formula that [7, Sec. 2.2]

$$c_\mu = \sum_\lambda K_{\lambda\mu}\Delta_\lambda, \tag{4.1}$$

where $K_{\lambda\mu}$ is the Kostka number and the sum runs over all partitions of n. We note that $K_{\lambda\lambda} = 1$ and that

$$K_{\lambda\mu} \neq 0 \text{ if and only if } \mu \trianglelefteq \lambda \text{ (i.e. } \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \text{ for all } i).$$

Proposition 4.1. *If T_X is nef and big, then X is Fano.*

Proof. Form the discussion above, it suffices to show

$$c_1^n = 0 \Rightarrow \Delta_{(1^n)} = 0.$$

By (4.1),

$$c_1^n = c_{(1^n)} = \sum_\lambda K_{\lambda(1^n)}\Delta_\lambda = 0.$$

Since $(1^n) \trianglelefteq \lambda$ for all partition λ, we have $K_{\lambda(1^n)} > 0$ and hence $\Delta_\lambda = 0$ for all λ. In particular, $\Delta_{(1^n)} = 0$. □

Remark 4.2. Proposition 4.1 provides new examples where D_S is not simple, generalizing the classical examples in [1, 15].

Inspired by the proof of Proposition 4.1, we prove the following theorem that was conjectured in [3] and was verified for $n = \dim X \leq 4$ [4].

Theorem 4.3. *Let X be a projective manifold with T_X nef. If $\chi(\mathcal{O}_X) \neq 0$, then X is Fano.*

Proof. It suffices to show that $c_1^n = 0$ implies $\chi(\mathcal{O}_X) = 0$. By Hirzebruch–Riemann–Roch [8, Corollary 15.2.2],

$$\chi(\mathcal{O}_X) = \int_X \mathrm{td}_n(T_X).$$

Since the nth Todd polynomial $\mathrm{td}_n(T_X)$ of T_X is a $\mathbb{Q}$-linear combination of the c_λ's [8, Example 3.2.4], we only have to show that $c_\lambda = 0$ for all $\lambda \in \Lambda_n$. Again, it follows from (4.1) that

$$[c_1^n = 0] \implies [\Delta_\lambda = 0 \text{ for all } \lambda \in \Lambda_n] \implies [c_\lambda = 0 \text{ for all } \lambda \in \Lambda_n]. \qquad \square$$

Example 4.4. We give some explicit examples to illustrate (4.1).

(1) When $n = 1$, $c_1 = \Delta_{(1)}$.
(2) When $n = 2$, write $\lambda_1 = (1, 1)$ and $\lambda_2 = (2, 0)$. The Segre classes

$$\Delta_{(1,1)} = c_1^2 - c_2 = c_{\lambda_1} - c_{\lambda_2} \quad \text{and} \quad \Delta_{(2,0)} = c_2 = c_{\lambda_2}.$$

We have

$$(c_{\lambda_1}, c_{\lambda_2}) = (\Delta_{\lambda_1}, \Delta_{\lambda_2}) \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(3) When $n = 3$, write $\lambda_1 = (1,1,1)$, $\lambda_2 = (2,1,0)$ and $\lambda_3 = (3,0,0)$. We have

$$(c_{\lambda_1}, c_{\lambda_2}, c_{\lambda_3}) = (\Delta_{\lambda_1}, \Delta_{\lambda_2}, \Delta_{\lambda_3}) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

(4) When $n = 4$, write $\lambda_1 = (1,1,1,1)$, $\lambda_2 = (2,1,1,0)$, $\lambda_3 = (2,2,0,0)$, $\lambda_4 = (3,1,0,0)$ and $\lambda_5 = (4,0,0,0)$. We have

$$(c_{\lambda_1}, c_{\lambda_2}, c_{\lambda_3}, c_{\lambda_4}, c_{\lambda_5}) = (\Delta_{\lambda_1}, \Delta_{\lambda_2}, \Delta_{\lambda_3}, \Delta_{\lambda_4}, \Delta_{\lambda_5}) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Conversely, we ask the following.

Question 4.5. If X is Fano with nef T_X, is it true that T_X is big?

Using the classification in [3], Question 4.5 can be verified for $n \leq 3$ by computing the Segre class $\Delta_{(1^n)}$ or quoting the known results.

(1) When $n = 1$, the only Fano curve is $\mathbb{P}^1$ and $T_{\mathbb{P}^1}$ is ample.
(2) When $n = 2$, the only Fano surfaces with nef T_X are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. These are toric varieties and hence have big tangent bundles.
(3) When $n = 3$, the only Fano threefolds with nef T_X are $\mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, Q_3, and $\mathbb{P}(T_{\mathbb{P}^2})$. The first three are toric varieties. The last two have big tangent bundles by [5, Example 1,2].

A stronger version of Question 4.5 (dropping the nefness assumption) has to do with D-simplicity of the section ring S (see Question 5.2).

5. Questions on D-Simplicity

Let Y be an affine algebraic variety over $\mathbb{C}$. When Y is smooth, it is well-known that its ring of differential operators D_Y is simple and Noetherian. The example in [1] shows that D_Y may not be simple nor Noetherian. On the other hand, it is shown in [16] that D_Y is simple if Y is a geometric invariant quotient of classical algebraic group acting on an affine space. It is asked in [16] whether the same result holds for other linear algebraic groups. See also Conjectures 1.1 and 1.2 of [22], in which the conjectures are proved in positive characteristics. Unfortunately, in view of Proposition 3.1, Theorem 1.2 is not helpful in finding counterexamples to this problem.

The results in [16] also suggest that D_Y might have nice properties when Y has only rational singularities. Although this is not true in general, it was proposed in [19] that Gorensteinness should play a role in answering this question. In fact, the main theorem in [19] shows that in positive characteristics a domain is strongly F-regular if and only if it is F-split and D-simple. In view of the correspondence between strong F-regularity and log terminal singularities and the fact that Gorenstein log terminal singularities are rational singularities, it is natural to ask the following question.

Question 5.1. Let Y be an affine variety over $\mathbb{C}$ that is Gorenstein. Is it true that Y has at worst rational singularities if and only if $\mathcal{O}_Y$ is a simple D_Y-module?

By [20, Proposition 6.2(2)], a normal projective variety is Fano and has rational singularities if and only if it admits a Gorenstein section ring with rational singularities. Therefore, Theorem 1.2 suggests the following question in Fano geometry.

Question 5.2. Does there exist a smooth Fano variety over $\mathbb{C}$ whose tangent bundle is not big?

A positive answer to Question 5.2 will provide a counterexample to the necessary part of Question 5.1.

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References

[1] I. N. Bernšteĭn, I. M. Gel′fand and S. I. Gel′fand, Differential operators on a cubic cone, *Uspehi Mat. Nauk* **27**(1)(163) (1972) 185–190, MR 0385159 (52 #6024).

[2] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge Studies in Advanced Mathematics, Vol. 39 (Cambridge University Press, Cambridge, 1993), MR MR1251956 (95h:13020).

[3] F. Campana and T. Peternell, Projective manifolds whose tangent bundles are numerically effective, *Math. Ann.* **289**(1) (1991) 169–187, MR 1087244 (91m:14061).

[4] F. Campana and T. Peternell, 4-folds with numerically effective tangent bundles and second Betti numbers greater than one, *Manuscripta Math.* **79**(3–4) (1993) 225–238, MR 1223018 (94e:14052).

[5] L. E. Solá Conde and J. A. Wiśniewski, On manifolds whose tangent bundle is big and 1-ample, *Proc. London Math. Soc.* (3) **89**(2) (2004) 273–290, MR 2078708 (2005c:14018).

[6] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies, Vol. 131 (Princeton University Press, Princeton, NJ, 1993), the William H. Roever Lectures in Geometry, MR MR1234037 (94g:14028).

[7] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts, Vol. 35 (Cambridge University Press, Cambridge, 1997), with applications to representation theory and geometry, MR 1464693 (99f:05119).

[8] W. Fulton, *Intersection Theory*, 2nd edn., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics], Vol. 2 (Springer-Verlag, Berlin, 1998), MR 1644323 (99d:14003).

[9] J. González, M. Hering, S. Payne and H. Süß, Cox rings and pseudoeffective cones of projectivized toric vector bundles, *Algebra Number Theory* **6**(5) (2012) 995–1017, MR 2968631.

[10] R. Hartshorne, Stable reflexive sheaves, *Math. Ann.* **254**(2) (1980) 121–176, MR 597077 (82b:14011).

[11] J.-C. Hsiao, D-module structure of local cohomology modules of toric algebras, *Trans. Amer. Math. Soc.* **364**(5) (2012) 2461–2478, MR 2888215.

[12] Y. Ishibashi, Remarks on a conjecture of Nakai, *J. Algebra* **95**(1) (1985) 31–45, MR 797652 (87b:13004).

[13] R. Lazarsfeld, *Positivity in Algebraic Geometry I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics], Vol. 48 (Springer-Verlag, Berlin, 2004), Classical setting: line bundles and linear series, MR 2095471 (2005k:14001a).

[14] R. Lazarsfeld, *Positivity in algebraic geometry II*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics], Vol. 49 (Springer-Verlag, Berlin, 2004), Positivity for vector bundles, and multiplier ideals, MR MR2095472 (2005k:14001b).

[15] T. Levasseur, *Opérateurs différentiels sur les surfaces munies d'une bonne* $\mathbf{C}^*$*-action*, Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), Lecture Notes in Mathematics, Vol. 1404 (Springer, Berlin, 1989), pp. 269–295, MR 1035229 (91g:14026).

[16] T. Levasseur and J. T. Stafford, Rings of differential operators on classical rings of invariants, *Mem. Amer. Math. Soc.* **81**(412) (1989) vi+117, MR 988083 (90i:17018).

[17] S. Mori, Projective manifolds with ample tangent bundles, *Ann. of Math.* (2) **110**(3) (1979) 593–606, MR 554387 (81j:14010).

[18] Y. Nakai, High order derivations. I, *Osaka J. Math.* **7** (1970) 1–27, MR 0263804 (41 #8404).

[19] K. E. Smith, The D-module structure of F-split rings, *Math. Res. Lett.* **2**(4) (1995) 377–386, MR 1355702 (96j:13024).

[20] K. E. Smith, Globally F-regular varieties: Applications to vanishing theorems for quotients of Fano varieties, *Michigan Math. J.* **48** (2000) 553–572, Dedicated to William Fulton on the occasion of his 60th birthday, MR 1786505 (2001k:13007).

[21] K. E. Smith, The multiplier ideal is a universal test ideal, *Comm. Algebra* **28**(12) (2000) 5915–5929, Special issue in honor of Robin Hartshorne, MR 1808611 (2002d:13008).

[22] K. E. Smith and M. Van den Bergh, Simplicity of rings of differential operators in prime characteristic, *Proc. London Math. Soc.* (3) **75**(1) (1997) 32–62, MR 1444312 (98d:16039).