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FIXED IDEALS OF CERTAIN CARTIER ALGEBRAS OVER *F*-SPLIT TORIC ALGEBRAS

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In this article, the ideals of an F-split toric algebra that are fixed by certain Cartier algebra are described, extending the result for normal toric algebras in the author's joint work with K. Schwede and W. Zhang.

Key Words: Cartier modules; Frobenius split; Toric algebras.

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1. INTRODUCTION

Let S be a toric algebra over a perfect field $\mathbb{K}$ of characteristic $p > 0$. Given a toric p^{-e}-linear map $\phi : S \to S$ (also called a toric near splitting), a monomial ideal $\mathfrak{a}$ of S, and a positive rational number t that has no p in the denominator, we are interested in the ideals I of S satisfying the condition

$$\sum_{n>0} \phi^n \left(\overline{\mathfrak{a}^{t(p^{ne}-1)}} \cdot I\right) = I, \tag{1.1}$$

where the sum runs through all $n \in \mathbb{N}$ such that $t(p^{ne}-1) \in \mathbb{N}$ and $\overline{\mathfrak{a}^{t(p^{ne}-1)}}$ is the integral closure of $\mathfrak{a}^{t(p^{ne}-1)}$ in S. These ideals are the so-called F-pure submodules of certain Cartier algebra $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$ in the sense of [1]. We call the ideals satisfying (1.1) $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed. When S is normal, $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed ideals are described in [4]. In particular, there are only finitely many of them. This provides a positive evidence to [1, Question 5.3]. Moreover, the $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed ideals of S are identified in [4] with certain multiplier-like ideals on the normal toric variety Spec$S$ arising from birational geometry.

The goal of this article is to generalize the description of $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed ideals in [4] to F-split toric algebras. To state the result, we need the following notation, which will be fixed throughout the article.

Notation. Let $\mathbb{K}$ be a perfect field of characteristic $p > 0$, and let Λ be a semigroup satisfying $\mathbb{Z}\Lambda = \mathbb{Z}^d = M$. We assume that 0 is the only invertible element

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in Λ. Denote $\sigma := \mathbb{R}_{\geq 0}\Lambda$ the real cone of Λ in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, and denote $\widetilde{\Lambda} := \sigma \cap M$ the saturation of Λ. For any face τ of σ, denote $\operatorname{int}\tau$ the relative interior of τ. Let $S := \mathbb{K}[\Lambda]$ be the semigroup algebra (toric algebra) generated by Λ over $\mathbb{K}$. Similarly, denote $\widetilde{S} = \mathbb{K}[\widetilde{\Lambda}]$.

Fix a number $e \in \mathbb{N}$, a p^{-e}-linear map (or a near splitting) on S is a map $\phi : S \to S$ satisfying $\phi(r^{p^e} s) = r\phi(s)$ for all $r, s \in S$. A toric near splitting is a near splitting $\phi_a : S \to S$ for some suitable $a \in \frac{1}{p^e}M$ such that for $\alpha \in \Lambda$

$$\phi_a(x^\alpha) = \begin{cases} x^{a+\frac{\alpha}{p^e}} & \text{if } a + \frac{\alpha}{p^e} \in M, \\ 0 & \text{otherwise.} \end{cases}$$

See Proposition 3.1 for a characterization of toric near splittings on toric algebras.

Fix a toric near splitting $\phi = \phi_a$, and set

$$\Gamma_{\phi,\mathfrak{a}^t} := \left(\frac{p^e a}{p^e - 1} + t \operatorname{Newt}\mathfrak{a}\right) \cap \Lambda.$$

Here, $\operatorname{Newt}\mathfrak{a}$ is defined to be the usual Newton polyhedra $\operatorname{Newt}(\mathfrak{a}\widetilde{S})$ in the normal case.

For $\alpha \in \Gamma_{\phi,\mathfrak{a}^t}$, consider

$$J_\alpha^{\phi,\mathfrak{a}^t} := \sum_{n \geq 0} \phi^n \left(\overline{\mathfrak{a}^{t(p^{ne}-1)}} \cdot \langle x^\alpha \rangle\right).$$

Since ϕ and $\mathfrak{a}^t$ are fixed, we simply write J_α for $J_\alpha^{\phi,\mathfrak{a}^t}$.

For each face $\tau \subseteq t \operatorname{Newt}\mathfrak{a}$, denote

$$\Gamma_{\phi,\tau} := \left(\frac{p^e a}{p^e - 1} + \operatorname{int}\tau\right) \cap \Lambda.$$

Let G_τ be the minimal subset of $\Gamma_{\phi,\tau}$ such that $\{x^\alpha \mid \alpha \in G_\tau\}$ generates the ideal $\langle x^\alpha \mid \alpha \in \Gamma_{\phi,\tau}\rangle$, and denote $G := \bigcup_{\tau \subseteq t \operatorname{Newt}\mathfrak{a}} G_\tau$.

Theorem 1.1. *Assume S is F-split. An ideal I of S is $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed if and only if $I = \sum_{\alpha \in H} J_\alpha$ for some subset $H \subseteq G$.*

When S is normal, we have $J_\alpha = J_{\alpha'}$ for any $\alpha, \alpha' \in G_\tau$ by [4, Proposition 3.4]. So Theorem 1.1 recovers [4, Theorem 3.5].

A simple example (Example 4.6) is computed to illustrate Theorem 1.1. We also determine the fixed ideals for some non-F-split toric algebra (Examples 4.7), showing that the situation is more subtle in the general case.

2. F-SPLIT TORIC ALGEBRAS

A ring R of prime characteristic p is called F-split if the Frobenius map $F : R \to R$ splits. The F-split toric algebras have been characterized before (see for example [2, Proposition 6.2]). For our purpose, we give an alternative characterization in terms of the toric language.

First, the following lemma is well known for a toric algebra $S = \mathbb{K}[\Lambda]$, .

Lemma 2.1 ([6, Proposition 3.4]). *There exists $b_i \in \widetilde{\Lambda}$ and faces τ_i of σ, $i = 1, 2, \ldots, m$, such that*

$$\widetilde{\Lambda} \backslash \Lambda = \bigsqcup_{i=1}^{m} [b_i + (\tau_i \cap \Lambda)] .$$

In particular, there exists $\alpha \in \Lambda \cap \operatorname{int} \sigma$ such that $\alpha + \widetilde{\Lambda} \subset \Lambda$.

The proof of Lemma 2.1 basically follows from the fact that the finite S-module $\widetilde{S}/S$ admits a finite filtration such that the quotient of each step in the filtration is isomorphic to S/P for some graded associated prime P of S.

Proposition 2.2. *A toric algebra $S = \mathbb{K}[\Lambda]$ is F-split if and only if*

$$p\left(\widetilde{\Lambda} \backslash \Lambda\right) \cap \Lambda = \emptyset.$$

Proof. First, suppose there exists an element $\beta \in \widetilde{\Lambda} \backslash \Lambda$ such that $p\beta \in \Lambda$. Choose $\alpha \in [\Lambda \cap \operatorname{int} \sigma]$ as in Lemma 2.1. Then $(\alpha + p\beta) + \widetilde{\Lambda} \subset \Lambda$. In particular, $\alpha + p\beta$ and $\alpha + (p+1)\beta$ are both in Λ. If S was F-split, there would be a splitting $\phi : S \to S$ of the Frobenius map. We have

$$x^{\alpha + p\beta} \phi(x^{p\beta}) = \phi(x^{p(\alpha + p\beta) + p\beta}) = x^{\alpha + (p+1)\beta} \phi(1) = x^{\alpha + (p+1)\beta}.$$

This implies $\phi(x^{p\beta}) = x^{\beta} \in S$ since S is a domain. But this contradicts the assumption that $\beta \notin \Lambda$.

Conversely, if $p\left(\widetilde{\Lambda} \backslash \Lambda\right) \cap \Lambda = \emptyset$, then for any $\beta \in \Lambda$ either $\frac{\beta}{p} \in \Lambda$ or $\frac{\beta}{p} \notin M$. So we have the well-defined splitting ϕ of the Frobenius map on S:

$$\phi(x^{\beta}) = \begin{cases} x^{\frac{\beta}{p}} & \text{if } \frac{\beta}{p} \in M, \\ 0 & \text{otherwise.} \end{cases}$$

□

Example 2.3. Consider $\Lambda = \mathbb{N}^2 \backslash \{(k, 0) \mid k \text{ is odd}\}$. Then $\mathbb{K}[\Lambda]$ is split when p is odd.

Corollary 2.4. *If S is F-split, then $\left(\widetilde{\Lambda} \backslash \Lambda\right) \cap \operatorname{int} \sigma = \emptyset$. In particular, the only one-dimensional F-split toric algebra is the polynomial ring of one variable.*

Proof. If there is an element $\beta \in \left(\widetilde{\Lambda} \backslash \Lambda\right) \cap \operatorname{int} \sigma$, then by Proposition 2.2

$$p^n \beta \in \left(\widetilde{\Lambda} \backslash \Lambda\right) \cap \operatorname{int} \sigma$$

for all $n \in \mathbb{N}$. This contradicts Lemma 2.1, which guarantees that $k\beta + \widetilde{\Lambda} \subset \Lambda$ for large $k \in \mathbb{N}$. □

Corollary 2.5. *Assume S is F-split. Suppose $\mathbf{v} \in \widetilde{\Lambda} \backslash \Lambda$ such that $\mathbf{v}$ is the primitive lattice point on the ray $\rho_{\mathbf{v}} := \mathbb{R}_{\geq 0}\mathbf{v}$. Then $\rho_{\mathbf{v}} \cap \Lambda = \{k(n_{\mathbf{v}}\mathbf{v}) | k \in \mathbb{N}\}$ for some $n_{\mathbf{v}} \in \mathbb{N}$ which is not divisible by p.*

Proof. First notice that some multiple of $\mathbf{v}$ is in Λ, so $\rho_{\mathbf{v}} \cap \Lambda \neq \emptyset$. Let $n_{\mathbf{v}}$ be the smallest positive integer n such that $n\mathbf{v} \in \Lambda$. This implies $n_{\mathbf{v}}$ is not divisible by p, for otherwise $n_{\mathbf{v}} = pk$ for some $k \in \mathbb{N}$. By Proposition 2.2,

$$p(k\mathbf{v}) = n_{\mathbf{v}}\mathbf{v} \in \Lambda \implies k\mathbf{v} \in \Lambda.$$

This contradicts the minimality of $n_{\mathbf{v}}$.

To complete the proof, assume that $0 < n_0 = n_{\mathbf{v}} < n_1 < n_2 < \ldots$ be all the integers satisfying $n_i\mathbf{v} \in \Lambda$. We will show by induction that each n_i is divisible by $n_{\mathbf{v}}$. Clearly, n_0 is divisible by $n_{\mathbf{v}}$. Suppose $n_0, \ldots, n_{k-1}$ is divisible by $n_{\mathbf{v}}$. Since p does not divide $n_{\mathbf{v}}$, there exists $0 \leq l \leq p-1$ such that $p|(ln_{\mathbf{v}} + n_k)$. By Proposition 2.2 again, $\frac{ln_{\mathbf{v}}+n_k}{p}\mathbf{v} \in \Lambda$. But $\frac{ln_{\mathbf{v}}+n_k}{p} = \frac{(p-1)n_k+n_k}{p} < n_k$ implies that $\frac{ln_{\mathbf{v}}+n_k}{p} = n_i$ for some $i < k$. Therefore, the induction hypothesis implies that $n_k = pn_i - ln_{\mathbf{v}}$ is divisible by $n_{\mathbf{v}}$. □

We will use the following lemma, whose proof is evident.

Lemma 2.6. *Let $n \in \mathbb{N}$, and let p be a prime number such that $p \nmid n$. Then for any $e \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that*

$$n \text{ divides } \sum_{i=0}^{l-1} p^{ei}.$$

Corollary 2.7. *Assume S is F-split. If $\lambda \in \widetilde{\Lambda}$, then for any $e \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that*

$$\left(\frac{p^{le}-1}{p^e-1}\right)\lambda = \left(\sum_{i=0}^{l-1} p^{ei}\right)\lambda \in \Lambda.$$

Proof. If $\lambda \in \Lambda$, simply take $l = 1$. Suppose $\lambda \notin \Lambda$. As in Corollary 2.5, let $\mathbf{v} \in \widetilde{\Lambda} \backslash \Lambda$ be the primitive lattice point on the ray $\mathbb{R}_{\geq 0}\lambda$, let $n_{\mathbf{v}} \in \mathbb{N}$ be the smallest positive integer such that $n_{\mathbf{v}}\mathbf{v} \in \Lambda$, and write $\lambda = n_\lambda \mathbf{v}$. Since $p \nmid n_{\mathbf{v}}$, by Lemma 2.6 there exists $k, l \in \mathbb{N}$ such that

$$kn_{\mathbf{v}} = \left(\sum_{i=0}^{l-1} p^{ei}\right).$$

Therefore,

$$\left(\sum_{i=0}^{l-1} p^{ei}\right)\lambda = kn_\lambda(n_{\mathbf{v}}\mathbf{v}) \in \Lambda.$$

□

3. TORIC NEAR SPLITTINGS

In this section, we determine all the toric near splittings on an F-split toric algebra S following [5].

Let ϕ_0 be the canonical splitting on $\mathbb{K}[M]$, namely for $\alpha \in M$,

$$\phi_0(x^\alpha) = \begin{cases} x^{\frac{\alpha}{p}} & \text{if } \frac{\alpha}{p} \in M, \\ 0 & \text{otherwise.} \end{cases}$$

For $e \in \mathbb{N}$ and $a \in \frac{1}{p^e}M$, consider the near splitting $\phi_a : \mathbb{K}[M] \to \mathbb{K}[M]$ defined by

$$\phi_a(\Box) = \phi_0^e(x^{p^e a} \cdot \Box).$$

We say ϕ_a is regular on S if $\phi_a(S) \subseteq S$ (i.e., ϕ_a induces a well-defined near splitting on S). In this case, we call ϕ_a a near splitting of S. It is observed in [5] that the ϕ_a's form a basis of the space of near splittings

$$\mathrm{Hom}_{\mathbb{K}[M]}(F_*^e\mathbb{K}[M], \mathbb{K}[M])$$

and that an element $\sum c_i\phi_{a_i} \in \mathrm{Hom}_{\mathbb{K}[M]}(F_*^e\mathbb{K}[M], \mathbb{K}[M])$, $c_i \in \mathbb{K}^\times$, lies in $\mathrm{Hom}_S(F_*^eS, S)$ if and only if each ϕ_{a_i} is regular on S. Here, for a module M over a ring R of positive characteristic p, we denote F_*^eM the additive group M equipped with an R-module structure via the R-action $r \cdot m := r^{p^e}m$. Notice that in [5], the monomials in F_*^eS are identified with the points in $\frac{1}{p^e}\Lambda$. We do not use that identification here.

We generalize Proposition 4.2 in [5] to general toric algebras.

Proposition 3.1. *For $a \in \frac{1}{p^e}M$, the map ϕ_a is regular on S if and only if*

$$a \in \bigcap_{\mathbf{v}_\rho} \left\{ v \in \frac{1}{p^e}M \,\middle|\, (v, \mathbf{v}_\rho) > -1 \right\} \setminus \bigcup_{\beta \in \widetilde{\Lambda}\setminus\Lambda} \left(\beta - \frac{1}{p^e}\Lambda\right).$$

Here the $\mathbf{v}_\rho$'s are the primitive lattice points on the rays of the dual cone $\sigma^\vee$ of $\sigma = \mathbb{R}_{\geq 0}\Lambda$.

Proof. We first prove the (if) part. Let

$$a \in \bigcap_{\mathbf{v}_\rho} \left\{ v \in \frac{1}{p^e}M \,\middle|\, (v, \mathbf{v}_\rho) > -1 \right\} \setminus \bigcup_{\beta \in \widetilde{\Lambda}\setminus\Lambda} \left(\beta - \frac{1}{p^e}\Lambda\right),$$

and suppose $\lambda \in \Lambda$ with $\frac{\lambda}{p^e} + a \in M$. We need to show that $\frac{\lambda}{p^e} + a \in \Lambda$ (i.e., $\phi_a(x^\lambda) = x^{\frac{\lambda}{p^e}+a} \in S$). Observe that

$$\left(\frac{\lambda}{p^e} + a, \mathbf{v}_\rho\right) > \left(\frac{\lambda}{p^e}, \mathbf{v}_\rho\right) - 1 \geq -1$$

for all $\mathbf{v}_\rho$ and hence $\frac{\lambda}{p^e} + a \in \widetilde{\Lambda}$. This implies $\frac{\lambda}{p^e} + a \in \Lambda$, since otherwise

$$a = \left(\frac{\lambda}{p^e} + a\right) - \frac{\lambda}{p^e} \in \left[\left(\frac{\lambda}{p^e} + a\right) - \frac{1}{p^e}\Lambda\right]$$

contradicting the assumption.

We now prove the (only if) part.

Suppose $(a, \mathbf{v}_{\rho_0}) \leq -1$ for some ρ_0. Using the notation in Lemma 2.1, we choose $b = b_i$ so that $\tau = \tau_i$ is the facet corresponding to ρ_0 and such that if $j \neq i$ is such that $\tau_j = \tau$ then $(b_j, \mathbf{v}_{\rho_0}) \leq (b, \mathbf{v}_{\rho_0})$. If there is no such b, just choose any $b \in \mathbb{Z}^d$ so that $(b, \mathbf{v}_{\rho_0}) = -1$. Now, choose $\beta \in [b + (\Lambda \cap \operatorname{int} \tau)] \subseteq (\mathbb{Z}^d \setminus \Lambda)$ so that $(\beta - a, \mathbf{v}_\rho) > (b_j, \mathbf{v}_\rho)$ for all $b_j \neq b$ and all $\rho \neq \rho_0$ (if no such b_j exists, we require $(\beta - a, \mathbf{v}_\rho) > 0$ for all $\rho \neq \rho_0$). Notice that $(\beta - a, \mathbf{v}_{\rho_0}) \geq (\beta, \mathbf{v}_{\rho_0}) + 1 = (b, \mathbf{v}_{\rho_0}) + 1$. By Lemma 2.1 and by the choice of b, we must have $\beta - a \in \frac{1}{p^e}\widetilde{\Lambda}$ and

$$[(\beta - a) + \sigma] \cap (\widetilde{\Lambda} \setminus \Lambda) = \emptyset.$$

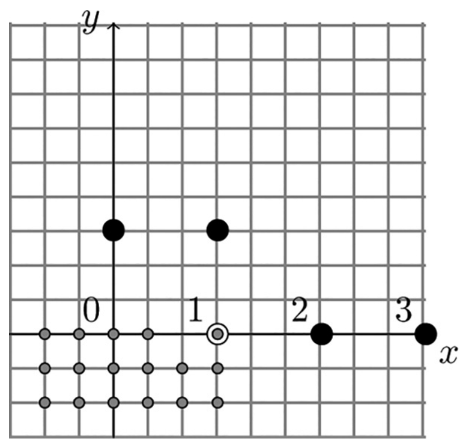
Therefore, ϕ_a is not regular since $p^e(\beta - a) \in \Lambda$ but $\phi_a(x^{p^e(\beta-a)}) = x^\beta \notin S$.

To complete the proof, we need to show that ϕ_a is regular implies

$$a \notin \bigcup_{\beta \in \widetilde{\Lambda} \setminus \Lambda} \left(\beta - \frac{1}{p^e}\Lambda\right).$$

Suppose $a \in \beta - \frac{1}{p^e}\Lambda$ for some $\beta \in \widetilde{\Lambda} \setminus \Lambda$. Then $p^e(\beta - a) \in \Lambda$. Since $\phi_a(x^{p^e(\beta-a)}) = x^\beta \notin S$, we see that ϕ_a is not regular. □

We illustrate Proposition 3.1 by the following figure. Consider the toric algebra generated by the four black dots $(0, 1), (1, 1), (2, 0), (3, 0)$ in characteristic $p = 3$. We have $\widetilde{\Lambda} \setminus \Lambda = \{(1, 0)\}$. The gridded area (excluding $x = -1$ and $y = -1$) indicates the candidates $a \in \bigcap_{\mathbf{v}_\rho} \left\{ v \in \frac{1}{p}M \,\middle|\, (v, \mathbf{v}_\rho) > -1 \right\}$ for ϕ_a to be a regular map.



The small gray dots indicate the points in $\bigcup_{\beta\in\widetilde{\Lambda}\setminus\Lambda}\left(\beta-\frac{1}{p}\Lambda\right)$ that should be removed from the candidate set.

4. $\mathscr{C}^{\phi,\overline{\mathfrak{a}^T}}$-FIXED IDEALS

In this section, fix an F-split toric algebra S, a monomial ideal $\mathfrak{a}$ of S, and a number $t \in \mathbb{Q}_{>0}$. Given an element $a \in \frac{1}{p^e}M$ such that the toric near splitting $\phi = \phi_a$ is regular on S, we have for each $\alpha \in \Lambda$ either $\frac{\alpha}{p^e} + a \in \Lambda$ or $\frac{\alpha}{p^e} + a \notin M$. We are going to characterize the $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed ideals.

To simplify the presentation, we introduce for any $n \in \mathbb{Z}$ and for any $\alpha \in M$ the element

$$\alpha_n := p^{ne}\left(\alpha - \frac{p^e a}{p^e-1}\right) + \frac{p^e a}{p^e-1}.$$

We have immediately that

$$\alpha_n - \alpha = (p^{ne}-1)\left(\alpha - \frac{p^e a}{p^e-1}\right). \tag{4.1}$$

Also, if $n \geq 0$, then $\alpha_n = (\alpha_n - \alpha) + \alpha \in M$, and one can also check that

$$\phi^n(x^{\alpha_n}) = x^{\alpha}. \tag{4.2}$$

Our goal is to describe the ideals $I \subseteq S$ satisfying (1.1)

$$\sum_{n>0}\phi^n\left(\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot I\right) = I,$$

where the sum runs through all $n \in \mathbb{N}$ such that $t(p^{ne}-1) \in \mathbb{N}$ and $\overline{\mathfrak{a}^{t(p^{ne}-1)}}$ is the integral closure of $\mathfrak{a}^{t(p^{ne}-1)}$ in S.

When S is normal, it is well known that the integral closure of a monomial ideal $J \subseteq S$ is

$$\overline{J} = \langle x^{\alpha} \mid \alpha \in \mathrm{Newt}(J) \cap \Lambda\rangle,$$

where $\mathrm{Newt}(J)$ is the convex hull of the set $\{\alpha \in M \mid x^{\alpha} \in J\}$. For general toric algebras S, define $\mathrm{Newt}(J) := \mathrm{Newt}(J\widetilde{S})$. The following lemma is valid without the F-split assumption on S.

Lemma 4.1. *Let I, J be monomial ideals of a toric algebra S. Then the following statements are true*:

(1) $\overline{J} = \langle x^{\alpha} \mid \alpha \in \mathrm{Newt}(J) \cap \Lambda\rangle$.
(2) $\mathrm{Newt}(I) + \mathrm{Newt}(J) = \mathrm{Newt}(IJ)$. *In particular*, $\overline{I^n} = \langle x^{\alpha} \mid \alpha \in (n\mathrm{Newt}(I) \cap \Lambda)\rangle$ *for any* $n \in \mathbb{N}$.
(3) *If* $I = \langle x^g \mid g \in \Gamma\rangle$, *then* $\mathrm{Newt}(I) = \mathrm{Conv}(\Gamma) + \sigma$ *where* $\mathrm{Conv}(\Gamma)$ *is the convex hull of* Γ.

Proof. When S is normal, these are proved in [4, Lemma 3.1]. All these statements extend readily to general S according to [3, Proposition 1.6.1] which states that if $R \subset S$ is an integral ring extension and I is an ideal of R then $\overline{IS} \cap R = \overline{I}$. □

One can also extend [4, Proposition 3.2] to F-split toric algebras.

Proposition 4.2. *If an ideal $I \subseteq S$ is $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed, then $I \subseteq I_{\phi,\mathfrak{a}^t} := \langle x^\alpha \mid \alpha \in \Gamma_{\phi,\mathfrak{a}^t} \rangle$ where*

$$\Gamma_{\phi,\mathfrak{a}^t} := \left(\frac{p^e a}{p^e - 1} + t \operatorname{Newt} \mathfrak{a} \right) \cap \Lambda.$$

Moreover, I is a monomial ideal.

Proof. The first statement is valid without the assumption that S is F-split. It follows from Lemma 4.1 and the first half of the proof of [4, Proposition 3.2]. Indeed, for any element

$$h = \sum c_j x^{\alpha_j} \in I \, (c_j \neq 0),$$

fix a term x^{α_j} and set $\alpha = \alpha_j$. Since $\sum_{n>0} \phi^n \left(\overline{\mathfrak{a}^{t(p^{ne}-1)}} \cdot I \right) = I$, there exists $n_1 > 0$, $\alpha' \in \Lambda$, and $\beta_1 \in t(p^{n_1 e} - 1) \operatorname{Newt}(\mathfrak{a}) \cap \Lambda$, such that $x^{\alpha'} \in I$ and $\alpha_{n_1} = \beta_1 + \alpha'$. Here α_{n_1} is defined as in (4.1) so that $\phi(x^{\alpha_{n_1}}) = x^\alpha$. Repeating the same process k times for α' in the place of α, for $2 \le i \le k$ we can find $n_i \in \mathbb{N}$, $\beta_i \in t(p^{n_i e} - 1) \operatorname{Newt}(\mathfrak{a}) \cap \Lambda$, and $\alpha^{(i)} \in \Lambda$ such that

$$\alpha_{\sum_{i=1}^k n_i} = \sum_{i=1}^{k-1} (\beta_i)_{n_{i+1}} + \beta_k + \alpha^{(k)}.$$

By definition (4.1), $(\beta_i)_{n_{i+1}} = p^{n_{i+1}e} \left(\beta_i - \frac{p^e a}{p^e - 1} \right) + \frac{p^e a}{p^e - 1}$, so

$$(\beta_i)_{n_{i+1}} \in \left[(1 - p^{n_{i+1}e}) \frac{p^e a}{p^e - 1} + t p^{n_{i+1}e} (p^{n_i e} - 1) \operatorname{Newt}(\mathfrak{a}) \right].$$

Hence, by Lemma 4.1(3)

$$\begin{aligned} \alpha_{\sum_{i=1}^k n_i} &= p^{\sum_{i=1}^k n_i} \alpha - \left(p^{\sum_{i=1}^k n_i} - 1 \right) \left(\frac{p^e a}{p^e - 1} \right) \\ &= \sum_{i=1}^{k-1} (\beta_i)_{n_{i+1}} + \beta_k + \alpha^{(k)} \\ &\in \left[\left(k - \sum_{i=1}^k p^{n_{i+1}e} \right) \frac{p^e a}{p^e - 1} + t \left(p^{\sum_{i=1}^k n_i e} - 1 \right) \operatorname{Newt}(\mathfrak{a}) \right]. \end{aligned}$$

Dividing by $p^{\sum_{i=1}^k n_i}$ and letting k go to infinity, we see that $\alpha \in \Gamma_{\phi,\mathfrak{a}^t}$ as desired.

For the second statement, since $(p^e-1)\left(\alpha-\frac{p^e a}{p^e-1}\right)\in[t(p^e-1)\operatorname{Newt}(\mathfrak{a})]\cap\widetilde{\Lambda}$ and since S is F-split, by Corollary 2.7 there exists $l\in\mathbb{N}$ such that $t(p^{le}-1)\in\mathbb{N}$ and

$$\alpha_l-\alpha \underset{(4.1)}{=} (p^{le}-1)\left(\alpha-\frac{p^e a}{p^e-1}\right)\in\left[t(p^{le}-1)\operatorname{Newt}(\mathfrak{a})\right]\cap\Lambda.$$

Therefore,

$$\phi^l\left(x^{\alpha_l-\alpha}\cdot h\right)\in\phi^l\left(\overline{\mathfrak{a}^{t(p^{le}-1)}}\cdot I\right)\subseteq I.$$

By (4.2),

$$\phi^l\left(x^{\alpha_l-\alpha}\cdot h\right)=\sum_j c_j^{\frac{1}{p^{le}}}x^{\alpha+\frac{\alpha_j-\alpha}{p^{le}}}=c_i^{\frac{1}{p^{le}}}x^{\alpha}.$$

Since l can be chosen arbitrarily large, we must have $x^\alpha\in I$. Moreover, since $\alpha=\alpha_i$ is arbitrary chosen, we conclude that I is a monomial ideal. □

We will often need the following lemma which is observed in the proof of Proposition 4.2.

Lemma 4.3. *For any $\alpha\in\Gamma_{\phi,\mathfrak{a}^t}$ (i.e. $x^\alpha\in I_{\phi,\mathfrak{a}^t}$), there exists $l\in\mathbb{N}$ such that $t(p^{le}-1)\in\mathbb{N}$ and*

$$x^{\alpha_l-\alpha}=x^{(p^{le}-1)\left(\alpha-\frac{p^e a}{p^e-1}\right)}\in\overline{\mathfrak{a}^{t(p^{le}-1)}}\subseteq S.$$

Proposition 4.4. *The ideal $I_{\phi,\mathfrak{a}^t}=\langle x^\alpha\mid\alpha\in\Gamma_{\phi,\mathfrak{a}^t}\rangle$ is $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed.*

Proof. We need to show

$$\sum_{n>0}\phi^n\left(\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot I_{\phi,\mathfrak{a}^t}\right)=I_{\phi,\mathfrak{a}^t}.$$

For the containment ($\subseteq$), fix any $n\in\mathbb{N}$ so that $t(p^{ne}-1)\in\mathbb{N}$, and let

$$x^\beta\in\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot I_{\phi,\mathfrak{a}^t}.$$

Then $\beta\in\left[t(p^{ne}-1)\operatorname{Newt}(\mathfrak{a})+\frac{p^e a}{p^e-1}+t\operatorname{Newt}\mathfrak{a}\right]=\left[\frac{p^e a}{p^e-1}+tp^{ne}\operatorname{Newt}\mathfrak{a}\right]$. So either $\phi^n(x^\beta)=0$ or

$$\phi^n(x^\beta)\underset{(4.2)}{=}x^{\beta_{-n}}=x^{p^{-ne}\left(\beta-\frac{p^e a}{p^e-1}\right)+\frac{p^e a}{p^e-1}},$$

where $\beta_{-n}\in\Gamma_{\phi,\mathfrak{a}^t}$. Therefore, $\phi^n(x^\beta)\in I_{\phi,\mathfrak{a}^t}$ as desired.

Conversely, let $x^\beta\in I_{\phi,\mathfrak{a}^t}$, and by Lemma 4.3 there exists $l\in\mathbb{N}$ such that

$$x^{\beta_l-\beta}\in\overline{\mathfrak{a}^{t(p^{le}-1)}}.$$

Therefore,

$$x^\beta = \phi^l(x^{\beta_l}) = \phi^l(x^{\beta_l-\beta}\cdot x^\beta) \in \phi^l\left(\overline{\mathfrak{a}^{t(p^{le}-1)}}\cdot I_{\phi,\mathfrak{a}^t}\right).$$

□

We now describe the basic $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed ideals.

As in the introduction, consider for each $\alpha \in \Gamma_{\phi,\mathfrak{a}^t}$ the ideal

$$J_\alpha = J_\alpha^{\phi,\mathfrak{a}^t} = \sum_{n\geq 0}\phi^n\left(\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot\langle x^\alpha\rangle\right)$$

It follows from Proposition 4.4 that $\langle x^\alpha\rangle \subseteq J_\alpha \subseteq I_{\phi,\mathfrak{a}^t}$.

Proposition 4.5.

(1) J_α *is* $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$*-fixed.*

(2) *If* I *is* $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$*-fixed, then for any* $x^\alpha \in I$ *we have* $J_\alpha \subseteq I$*. In particular, if* $I = \langle x^{\alpha_1},\ldots,x^{\alpha_k}\rangle$ *is* $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$*-fixed, then* $I = \sum_{i=1}^k J_{\alpha_i}$*.*

Proof. (1) By construction of J_α, we have

$$\sum_{n>0}\phi^n\left(\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot J_\alpha\right) \subseteq J_\alpha.$$

Conversely, let $x^\beta \in \langle x^\alpha\rangle$. By Lemma 4.3, there exists $l \in \mathbb{N}$ such that $t(p^{le}-1) \in \mathbb{N}$ and $x^{\beta_l-\beta} \in \overline{\mathfrak{a}^{t(p^{le}-1)}}$. So

$$x^\beta \underset{(4.2)}{=} \phi^l(x^{\beta_l-\beta}\cdot x^\beta) \in \phi^l\left(\overline{\mathfrak{a}^{t(p^{le}-1)}}\cdot\langle x^\alpha\rangle\right) \subseteq \phi^l\left(\overline{\mathfrak{a}^{t(p^{le}-1)}}\cdot J_\alpha\right).$$

Therefore,

$$J_\alpha = \left[\langle x^\alpha\rangle + \sum_{n>0}\phi^n\left(\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot\langle x^\alpha\rangle\right)\right] \subseteq \sum_{n>0}\phi^n\left(\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot J_\alpha\right).$$

(2) For the first statement, observe again that

$$J_\alpha = \sum_{n\geq 0}\phi^n\left(\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot\langle x^\alpha\rangle\right) \subseteq \sum_{n\geq 0}\phi^n\left(\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot I\right) = I.$$

The second statement follows immediately. □

Now, we are ready to describe all $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed ideals.

Recall that for each face $\tau \subseteq t\,\mathrm{Newt}\,\mathfrak{a}$,

$$\Gamma_{\phi,\tau} := \left(\frac{p^e a}{p^e-1} + \mathrm{int}\,\tau\right) \cap \Lambda,$$

G_τ is the minimal subset of $\Gamma_{\phi,\tau}$ such that $\{x^\alpha \mid \alpha \in G_\tau\}$ generates the ideal $\langle x^\alpha \mid \alpha \in \Gamma_{\phi,\tau}\rangle$, and $G := \bigcup_{\tau\subseteq t\,\mathrm{Newt}\,\mathfrak{a}} G_\tau$.

Proof of Theorem 1.1. The (if) part follows from Proposition 4.5(1) and the fact that sums of $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed ideals are $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed.

Conversely, if I is $\mathscr{C}^{\phi,\overline{\mathfrak{a}^t}}$-fixed, then by Proposition 4.2 I is a monomial ideal satisfying $I \subseteq I_{\phi,\mathfrak{a}^t}$. Let

$$H = \{\alpha \in G \mid x^\alpha \in I\}.$$

Then $I \supseteq \sum_{\alpha \in H} J_\alpha$ by Proposition 4.5(2). We claim that

$$H \neq \emptyset \quad \text{and} \quad I = \sum_{\alpha \in H} J_\alpha.$$

Let $x^\beta \in I$. By Proposition 4.2, there is a unique face $\tau \subseteq t\,\mathrm{Newt}(\mathfrak{a})$ such that $\beta \in \Gamma_{\phi,\tau}$. So there exists an $\alpha \in G_\tau$ such that $\beta \in \alpha + \Lambda$, namely $x^\beta \in \langle x^\alpha \rangle \subseteq J_\alpha$. It suffices to prove that $\alpha \in H$.

If $\alpha = \beta$, there is nothing to do. Suppose $\alpha \neq \beta$, and let δ be the face of σ such that $\beta - \alpha \in (\mathrm{int}\,\delta) \cap \Lambda$. On each ray (one-dimensional face) of δ, choose an element in Λ. Denote these elements by $\gamma_1, \ldots, \gamma_k$. Then there exists $l \in \mathbb{N}$ such that

$$l(\beta - \alpha) = \sum_{i=1}^{k} n_i \gamma_i, \ \text{ for some } n_i \in \mathbb{N}.$$

On the other hand, since

$$\alpha_1 - \alpha = (p^e - 1)\left(\alpha - \frac{p^e a}{p^e - 1}\right) \in \widetilde{\Lambda},$$

we must have $\alpha_1 - \alpha \in (\mathrm{int}\,\delta') \cap \widetilde{\Lambda}$ for some unique face δ' of σ. Note that if $\tau = t\,\mathrm{Newt}\,\mathfrak{a}$, then $\alpha_1 - \alpha \in \mathrm{int}\,\sigma$ and hence $\delta' = \sigma \supseteq \delta$. Also, if $\tau \subsetneq t\,\mathrm{Newt}\,\mathfrak{a}$, then

$$0 \neq \beta - \alpha \in [\Lambda \cap (\mathrm{int}\,\tau - \mathrm{int}\,\tau) \cap \mathrm{int}\,\delta]$$

implies that τ contains a translation of δ. In this case, since $\alpha \in \Gamma_{\phi,\tau}$, we still have $\delta' \supseteq \delta$. So we may choose an element in Λ on each ray of δ' that is not in δ (if exists). Denote these elements $\gamma_{k+1}, \ldots, \gamma_{k'}$ $(k \leq k')$. Write

$$\alpha_1 - \alpha = \sum_{i=1}^{k'} q_i \gamma_i \quad \text{for some } q_i \in \mathbb{Q}_{>0}.$$

Choose $u, v \in \mathbb{N}$ such that $p \nmid v$ and that $p^u v q_i \in \mathbb{N}$ for $i = 1, \ldots, k'$. By Lemma 2.6, there exists $n \in \mathbb{N}$ such that the following statements hold:

(1) $t(p^{ne} - 1) \in \mathbb{N}$;
(2) v devides $\frac{p^{ne}-1}{p^e-1}$;
(3) $\left(\frac{p^{ne}-1}{p^e-1}\right) p^u q_i \geq p^u n_i$, for $i = 1, \ldots, k$.

In particular, by (4.1)

$$\begin{aligned}
p^u[(\alpha_n-\alpha)-l(\beta-\alpha)] &= \left[\left(\frac{p^{ne}-1}{p^e-1}\right)p^u(\alpha_1-\alpha)-p^u l(\beta-\alpha)\right]\\
&= \sum_{i=1}^{k}\left[\left(\frac{p^{ne}-1}{p^e-1}\right)p^u q_i - p^u n_i\right]\gamma_i + \sum_{i=k+1}^{k'}\left[\left(\frac{p^{ne}-1}{p^e-1}\right)p^u q_i\right]\gamma_i\\
&\in \Lambda.
\end{aligned}$$

Since S is F-split, we have $(\alpha_n-\alpha)-l(\beta-\alpha)\in\Lambda$ by Proposition 2.2. Hence

$$\alpha_n-\beta\in[(l-1)(\beta-\alpha)+\Lambda]\subseteq\Lambda.$$

Notice also that, since τ contains a translation of δ, for l sufficiently large

$$\begin{aligned}
\alpha_n-\beta=(\alpha_n-\alpha)-(\beta-\alpha) &= (p^{ne}-1)\left(\alpha-\frac{p^e a}{p^e-1}\right)-\sum_{i=1}^{k}\frac{n_i}{l}\gamma_i\\
&\in (p^{ne}-1)\operatorname{int}\tau-\sum_{i=1}^{k}\frac{n_i}{l}\gamma_i\\
&\subseteq (p^{ne}-1)\operatorname{int}\tau\\
&\subseteq t(p^{ne}-1)\operatorname{Newt}\mathfrak{a}.
\end{aligned}$$

Therefore, we have $x^{\alpha_n-\beta}\in\overline{\mathfrak{a}^{t(p^{ne}-1)}}$ and

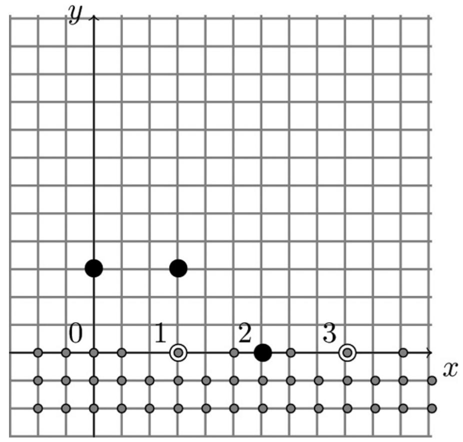
$$x^{\alpha}=\phi(x^{\alpha_n})=\phi(x^{\alpha_n-\beta}\cdot x^{\beta})\in\phi\left(\overline{\mathfrak{a}^{t(p^{ne}-1)}}\cdot I\right)\subseteq I$$

as desired. □

Example 4.6. As in Example 2.3, consider $S=\mathbb{K}[\Lambda]$ where

$$\Lambda=\mathbb{N}^2\backslash\{(k,0)\mid k\text{ is odd}\}.$$

Assume $\operatorname{char}\mathbb{K}=p=3$ and $e=1$. Again, the gray dots in the following figure indicate the points a with ϕ_a not regular.

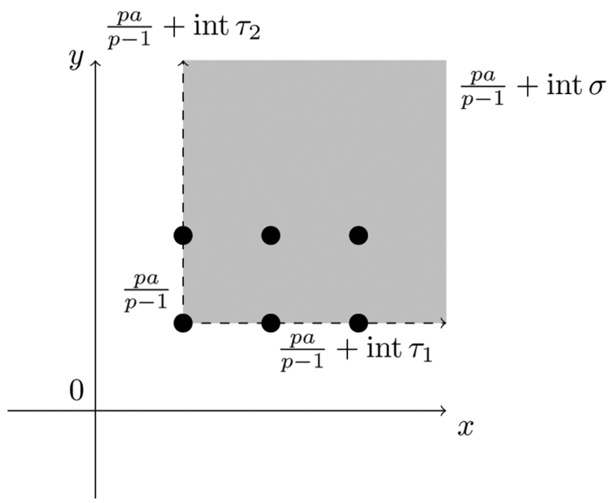


Consider $a = (\frac{2}{3}, \frac{2}{3})$. Then $\frac{pa}{p-1} = (1, 1)$ and

$$G = \bigcup_{\tau \subseteq \sigma} G_\tau = \{(1,1), (1,2), (2,1), (3,1), (2,2), (3,2)\}.$$

The following figure illustrates the points in G as well as the sets $\frac{pa}{p-1} + \operatorname{int}\tau$ for $\tau \subseteq \sigma$.
By Theorem 1.1, the nontrivial fixed ideals are as follows:

$$J_{(1,1)} = \langle xy \rangle, J_{(2,1)} = \langle x^2y \rangle, J_{(3,1)} = \langle x^3y, x^2y^2 \rangle,$$
$$J_{(1,2)} = \langle xy^2, x^2y^2 \rangle, J_{(2,2)} = J_{(3,2)} = \langle x^2y^2, x^3y^2 \rangle = \text{smallest},$$
$$J_{(1,1)} + J_{(2,1)} = \langle xy, x^2y \rangle = \text{largest},$$
$$J_{(2,1)} + J_{(3,1)} = \langle x^2y, x^3y \rangle,$$
$$J_{(2,1)} + J_{(1,2)} = \langle x^2y, xy^2 \rangle,$$
$$J_{(1,2)} + J_{(3,1)} = \langle xy^2, x^2y^2, x^3y \rangle,$$
$$J_{(2,1)} + J_{(3,1)} + J_{(1,2)} = \langle x^2y, x^3y, xy^2 \rangle.$$



Observe that since $\tau_1 \cap \left(\widetilde{\Lambda} \setminus \Lambda\right) \neq \emptyset$, the generators in G_{τ_1} produce multiple fixed ideals, $J_{(2,1)}$ and $J_{(3,1)}$, whereas the generators in G_σ produce only one fixed ideal, $J_{(2,2)} = J_{(3,2)}$.

Similarly, one can easily verify the following table.

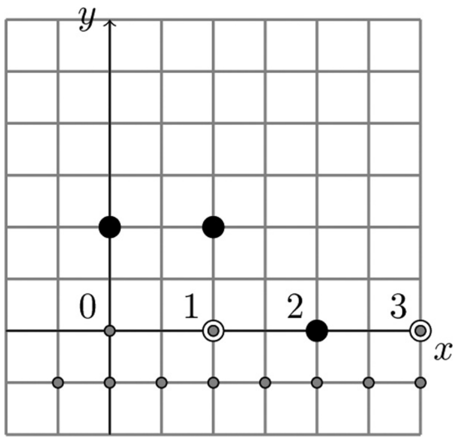
a	$\frac{pa}{p-1}$	Nontrivial ϕ_a fixed ideals
$(\frac{1}{3}, \frac{1}{3})$	$(\frac{1}{2}, \frac{1}{2})$	$\langle xy, x^2y \rangle$
$(\frac{2}{3}, \frac{1}{3})$	$(1, \frac{1}{2})$	$\langle xy, x^2y \rangle, \langle x^2y, x^3y \rangle$
$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{2}, 1)$	$\langle xy, x^2y \rangle, \langle xy^2, x^2y^2 \rangle$
$(\frac{2}{3}, 0)$	$(1, 0)$	$\langle x^2 \rangle, \langle xy, x^2y \rangle, \langle x^2y, x^3y \rangle$

In the following examples, we will see that many conclusions in this section are false for non-F-split toric algebras.

Example 4.7. Again, by Example 2.3 the toric algebra in characteristic $p = 2$ generated by $\Lambda = \mathbb{N}^2 \backslash \{(k, 0) \mid k \text{ is odd}\}$ is not F-split. By Proposition 3.1, the set

$$\left\{(u, v) \in \frac{1}{2}M \,\middle|\, u > -1, v > 0\right\} \bigcup \left\{\left(u - \frac{1}{2}, 0\right) \,\middle|\, u \geq 0\right\}$$

consists of all a with ϕ_a regular. Notice that in the following figure the points $(2k, 0) \in \Lambda$, $k \geq 0$, are corresponding to nonregular maps. This does not happen for F-split toric algebras.



We consider the following cases.

(1) $a = \left(-\frac{1}{2}, 0\right)$, $\frac{pa}{p-1} = (-1, 0)$. Observe that $x^{2k} \notin \phi_a(S)$ for all $k \geq 0$. In particular, as a generator in G, x^{2k} does not produce any fixed ideals. Also, even though $\Lambda \subseteq \frac{pa}{p-1} + \sigma$, the toric algebra S is not fixed by ϕ_a. The only nontrivial ϕ_a-fixed ideal is $\langle y, xy \rangle$.

(2) $a = \left(\frac{1}{2}, 0\right)$, $\frac{pa}{p-1} = (1, 0)$. Again, $x^{2k} \notin \phi_a(S)$ for all $k \geq 0$ and the only nontrivial ϕ_a-fixed ideals are $\langle xy, x^2y \rangle$, $\langle x^2y, x^3y \rangle$.

(3) $a = \left(-\frac{1}{2}, 1\right)$, $\frac{pa}{p-1} = (-1, 2)$. The set of generators is $G = \{(0, 2), (1, 2), (0, 3), (1, 3)\}$. One can easily check that $J_{(1,2)} = \langle y, xy^2 \rangle$ and $J_{(0,3)} = J_{(1,3)} = \langle y^3, xy^3 \rangle$ are ϕ_a-fixed ideals, but $J_{(0,2)} = \langle y^2 \rangle$ is not fixed by ϕ_a.

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