# MULTIGRADED HILBERT SCHEMES PARAMETRIZING IDEALS IN THE WEYL ALGEBRA

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ABSTRACT. Results of Haiman and Sturmfels **[HS04]** on multigraded Hilbert schemes are used to establish a quasi-projective scheme which parametrizes all left homogeneous ideals in the Weyl algebra having a fixed Hilbert function with respect to a given grading by an abelian group.

**1. Introduction.** Let $S = k[x_1, \dots, x_n]$ be the polynomial algebra over a commutative ring $k$. The monomials $x^u$ in $S$ are identified with their exponents $u \in \mathbb{N}^n$. A grading of $S$ by an abelian group $A$ is a semigroup homomorphism $\deg : \mathbb{N}^n \to A$. We may assume that $A$ is generated by $\deg(x_i)$ for $i = 1, \dots, n$. For $a \in A$, let $S_a$ be the $k$-span of the monomial $x^u$ with $\deg(u) = a$. We have the decomposition $S = \bigoplus_{a \in A} S_a$ that satisfies $S_a \cdot S_b \subseteq S_{a+b}$. An admissible ideal in $S$ is a homogeneous ideal $I$ with the property that $(S/I)_a = S_a/I_a$ is a locally free $k$-module of finite rank (constant on $\operatorname{Spec} k$) for all $a \in A$. The Hilbert function of an admissible ideal $I$ is a map $h_I : A \to \mathbb{N}$ defined by $h_I(a) := \operatorname{rank}_k(S_a/I_a)$.

Fix any function $h : A \to \mathbb{N}$, Haiman and Sturmfels construct in **[HS04]** a scheme $H_S^h$ over $k$ (called the multigraded Hilbert scheme) which parametrizes all admissible ideals $I$ in $S$ with Hilbert function $h_I = h$. As discussed in **[HS04]**, their results recover many special cases, including Hilbert schemes of points in affine space, toric Hilbert schemes, Hilbert schemes of abelian groups orbits, and Grothendieck Hilbert schemes. It is also mentioned in [**HS04,** section 6.2] that their results can be applied to the universal enveloping algebra of an $A$-graded Lie algebra. The purpose of this note is to verify this claim for the special case of the Weyl algebra $W = k\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$.

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In order to have a well-defined degree function on the set $\mathcal{B} = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$ of all monomials in $W$, we assume that our ground ring $k$ is an integral domain of characteristic 0. By Proposition 2.1 of **[Cou95,** Chapter 1] (the proof works for any integral domain $k$), the set $\mathcal{B}$ forms a $k$-basis for $W$. In general, this does not hold in the non-domain case. For example, if $k = \mathbb{Z}[t]/\langle 2t\rangle$, then $t\partial^2 \in k\langle x, \partial\rangle$ acts as the zero operator on $k[x]$. On the other hand, in view of the relations $\partial_i x_i - x_i \partial_i = 1$ in $W$ one quickly notices that we must have $\deg(x_i) = -\deg(\partial_i)$. Therefore, any $A$-grading $\deg : \mathbb{N}^n \to A$ on $S$ extends to an $A$-grading $\deg : \mathcal{B} \to A$ on $W$ by $\deg(x^\alpha \partial^\beta) = \deg(\alpha) - \deg(\beta)$. We have the decomposition $W = \bigoplus_{a \in A} W_a$ satisfying $W_a \cdot W_b \subseteq W_{a+b}$, where $W_a$ is the $k$-span of the monomials in $\mathcal{B}$ with degree $a$.

Just like the case of polynomial algebras, we call a homogeneous left ideal $I$ of $W$ admissible if $(W/I)_a = W_a/I_a$ is a locally free $k$-module of finite rank (constant on $\operatorname{Spec} k$) for all $a \in A$. Notice that the Hilbert function $h_I : A \to \mathbb{N}$ of an admissible ideal $I$ in $W$ defined by $h_I(a) = \operatorname{rank}_k(W_a/I_a)$ can not have finite support. This follows from the fact that there is no left ideal of $W$ with finite co-rank over $k$. Indeed, if $\operatorname{rank}(W/I)$ was finite, then the two $k$-linear maps $\phi_x$ and $\phi_\partial$ on $W/I$ induced by multiplications of $x$ and $\partial$ respectively would satisfy the equality $\phi_\partial \phi_x - \phi_x \phi_\partial = \mathrm{id}_{W/I}$, which is not possible by comparing the traces of the linear maps from both sides.

Our goal is to prove the following analog of **[HS04,** Theorem 1.1].

**Theorem 1.1.** *Given a Hilbert function $h : A \to \mathbb{N}$, there exists a quasi-projective scheme over $k$ that represents the Hilbert functor $H_W^h : \underline{k\text{-Alg}} \to \underline{\text{Set}}$ where for a $k$-algebra $R$ the set $H_W^h(R)$ consists of homogeneous ideals $I \subseteq R \otimes_k W$ such that $(R \otimes_k W_a)/I_a$ is a locally free $R$-module of rank $h(a)$ for every $a \in A$.*

We will recall in section 4 the techniques from **[HS04]** that are needed in the proof of Theorem 1.1. Roughly speaking, we first show that for any finite set of degrees $D$ the Hilbert functor $H_{W_D}^h$ is represented by a quasi-projective scheme that is a closed subscheme of a certain relative Grassmann scheme. Here, the $k$-module $W_D = \bigoplus_{a \in D} W_a$ and by abusing notation the restriction of the Hilbert function $h : A \to \mathbb{N}$ to $D$ is also denoted by $h$. Then we specify a special finite set $D$ such that $H^h_W$ is a subfunctor of $H^h_{W_D}$ represented by a closed subscheme of $H^h_{W_D}$.

Although the strategy of proving Theorem 1.1 is very similar to the polynomial algebra case, there are still several issues that require some modifications. For example, the key feature that makes these mechanisms work for the multigraded Hilbert scheme $H^h_S$ is the nice behaviors of monomial ideals in $S$ (*e.g.* the fact that antichains of monomial ideals in $S$ are finite [**Mac01**] is essential in the construction of $H^h_S$). We will see in section 2 that monomial ideals in $W$ do not have the expected behaviors in general. In particular, the naive generalization of Gröber basis theory to the Weyl algebra does not work very well. For example, the ideal $\langle \partial^2, x\partial - 1\rangle$ and its naive initial ideal $\langle \partial^2, x\partial\rangle = \langle \partial\rangle$ in $W = k\langle x, \partial\rangle$ do not have the same Hilbert function. To get around this, we consider the initial ideal of a left ideal in $W$ as a monomial ideal in the associated graded algebra $\operatorname{gr} W$ (which is a polynomial algebra) and utilize the Gröbner basis theory for the Weyl algebra developed in [**SST00**]. Basic facts about the Gröbner basis theory for $W$ will be reviewed in section 3. Finally, the proof of Theorem 1.1 will be elaborated in section 5.

The same results in this paper for right ideals can be achieved in similar manners.

**2. Monomial ideals in the Weyl algebra.** Let $k$ be an integral domain of characteristic 0 and let $W = k\langle x, \partial\rangle = k\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ be the $n$th Weyl algebra. Many classical facts about the Weyl algebra are proved under the assumption that $k$ is a field (see *e.g.* [**Cou95**]). This has an advantage that $W$ is left and right Noetherian. For our purpose, we will not make this assumption and many classical properties of the Weyl algebra extend to this more general setting. For examples, by [**Cou95,** Chapter 1, Proposition 2.1] (whose proof works for integral domain $k$) the set

$$\mathcal{B} = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$$

forms a $k$-basis for $W$, where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. The unique expression of an element $\delta$ of $W$ in terms of this $k$-basis $\mathcal{B}$ is called the canonical form of $\delta$. In this paper, elements in $\mathcal{B}$ are the only monomials of the Weyl algebra $W$ and a product of monomials in $W$ may not be a monomial. Also, the total degree of the monomial $x^\alpha \partial^\beta$

is $|\alpha| + |\beta|$ and the total degree of an element in $W$ is defined as the total degree of its leading monomials. The total degrees of elements of $W$ induce the Bernstein filtration on $W$, whose associated graded ring $\operatorname{gr} W = k[x, \xi] = k[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ is the polynomial algebra of $2n$ variables over $k$. Moreover, by considering the isomorphism of free $k$-modules $\Psi : k[x, \xi] \to W = k\langle x, \partial\rangle$ that sends $x^\alpha \xi^\beta$ to $x^\alpha \partial^\beta$, we have the following Leibniz formula that is helpful for the multiplication of elements in $W$. Note that in the formula of Proposition 2.1 the denominator $k_1! \cdots k_n!$ is used only to obtain a nice expression, we will never need to find the inverse of elements in the domain $k$.

**Proposition 2.1.** **[SST00,** Theorem 1.1.1] *For any two polynomials $f$ and $g$ in $k[x, \xi]$, we have*

$$\Psi(f) \cdot \Psi(g) = \sum_{k_1,\ldots,k_n \geq 0} \frac{1}{k_1! \cdots k_n!} \Psi\left(\frac{\partial^k f}{\partial \xi^k} \cdot \frac{\partial^k g}{\partial x^k}\right).$$

In particular, we have a convenient formula to multiply two monomials.

**Corollary 2.2.** *Let $x^\alpha \partial^\beta, x^{\alpha'} \partial^{\beta'}$ be monomials in $W$. We have*

$$(x^\alpha \partial^\beta) \cdot (x^{\alpha'} \partial^{\beta'}) = \sum_{k_1,\ldots,k_n \geq 0} \left(\prod_{i=1}^n k_i! \binom{\beta_i}{k_i} \binom{\alpha'_i}{k_i} x^{\alpha_i + \alpha'_i - k_i} \partial^{\beta_i + \beta'_i - k_i}\right).$$

A left ideal in $W$ is called a left monomial ideal if it is generated by monomials. Unlike the monomial ideals in a polynomial algebra, it can happen that an element in a left monomial ideal $I$ is a sum of monomials that are not in $I$. For example, in the first Weyl algebra $W = k\langle x, \partial\rangle$, the element $\partial x = x\partial + 1$ is in the principal ideal $I = Wx$ generated by $x$, but the identity 1 (hence $x\partial$) is not in $I$ by considering the total degrees of elements in $W$. Moreover, there exists an infinite antichain of monomial ideals in $W$ (see Example 2.6). So the direct generalization of the main theorem in **[Mac01]** does not hold for the Weyl algebra. Nonetheless, we still have the following analog of Dickson's lemma for monomial ideals in polynomial algebras.

**Proposition 2.3.** *Every left monomial ideal of $W$ is generated by finitely many monomials.*

*Proof.* Let $I$ be a left monomial ideal of $W$. By passing to the associated graded algebra with respect to the Bernstein filtration, one observes that the ideal $\operatorname{gr} I$ is a monomial ideal of $\operatorname{gr} W$. By Dickson's lemma, $\operatorname{gr} I$ is finitely generated by monomials of degrees $\leq m$ for some $m$. Standard arguments about filtered algebras (see, for example, the proof of [**Cou95,** Theorem 8.2.3]) show that $I$ is generated by elements with total degree $\leq m$, say $f_1, \dots, f_t$. We only need finitely many monomials in $I$ to generate $f_1, \dots, f_t$, so $I$ is in fact generated by finitely many monomials. $\square$

**Example 2.4.** Every left monomial ideal in the first Weyl algebra $W = k\langle x, \partial \rangle$ is principally generated by one monomial. To see this, suppose $I$ is a left monomial ideal of $W$ and assume that $x^\alpha \partial^\beta \in I$. Observe that by Corollary 2.2, we have

(i) $(x\partial)(x^\alpha \partial^\beta) = x^{\alpha+1}\partial^{\beta+1} + \alpha x^\alpha \partial^\beta$, so $x^{\alpha+1}\partial^{\beta+1} \in I$ and hence $x^{\alpha+s}\partial^{\beta+t} \in I$ for all $0 \leq t \leq s$;

(ii) for $\alpha \geq 1$, $\partial(x^\alpha \partial^\beta) = x^\alpha \partial^{\beta+1} + a x^{\alpha-1}\partial^\beta$, so $x^\alpha \partial^{\beta+1} \in I$ if and only if $x^{\alpha-1}\partial^\beta \in I$.

It suffices to show that for any two monomials $x^\alpha \partial^\beta$ and $x^{\alpha'}\partial^{\beta'}$ in $I$ there exists a monomial $x^{\alpha''}\partial^{\beta''} \in I$ such that $x^\alpha \partial^\beta, x^{\alpha'}\partial^{\beta'} \in W x^{\alpha''}\partial^{\beta''}$. We may assume by symmetry that $\beta' \geq \beta$ and consider only the following cases.

(a) $[0 \leq \beta' - \beta \leq \alpha' - \alpha]$. In this case, $x^{\alpha'}\partial^{\beta'} = x^{\alpha+(\alpha'-\alpha)}\partial^{\beta+(\beta'-\beta)} \in W x^\alpha \partial^\beta$ by (i), so we simply take $(\alpha, \beta) = (\alpha'', \beta'')$.

(b) $[\alpha' - \alpha < \beta' - \beta]$. In this case, take $\beta'' = \beta$ and

$$\alpha'' = \begin{cases} 0 & \text{if } \alpha' - \beta' + \beta \leq 0; \\ \alpha' - \beta' + \beta & \text{otherwise.} \end{cases}$$

It follows from (i) that $x^\alpha \partial^\beta, x^{\alpha'}\partial^{\beta'} \in W x^{\alpha''}\partial^{\beta''}$. On the other hand, the Leibniz formula in (ii) also shows that if $x^{\alpha''+1}\partial^{\beta''}$ and $x^{\alpha''+1}\partial^{\beta''+1}$ are both in $I$, then $x^{\alpha''}\partial^{\beta''} \in I$. Hence, if there exists $m \in \mathbb{N}$ such that $\{x^{\alpha''+m}\partial^{\beta''+i} \mid 0 \leq i \leq m\} \subset I$, then $x^{\alpha''}\partial^{\beta''} \in I$. The proof of this example is completed by taking $m = \alpha + \beta' - \beta - \alpha''$. Indeed, for $\beta' - \beta \leq i \leq m$,

$$0 \leq (\beta'' + i) - \beta' \leq m + (\beta - \beta') \leq m + (\alpha'' - \alpha') = (\alpha'' + m) - \alpha',$$

so by (i), $\{x^{\alpha''+m}\partial^{\beta''+i} \mid \beta' - \beta \leq i \leq m\} \subset Wx^{\alpha'}\partial^{\beta'} \subset I$. Also, for $0 \leq i < \beta' - \beta$,

$$0 \leq i = (\beta'' + i) - \beta < \beta' - \beta = (\alpha'' + m) - \alpha,$$

so by (i) again, $\{x^{\alpha''+m}\partial^{\beta''+i} \mid 0 \leq i < \beta' - \beta\} \subset Wx^{\alpha}\partial^{\beta} \subset I$.

The observations in Example 2.4 also imply the following lemma.

**Lemma 2.5.** *In the first Weyl algebra $W$, if $x^{\alpha'}\partial^{\beta'} \in Wx^{\alpha}\partial^{\beta}$ with $\alpha \geq 1$, then*

$$[(\alpha', \beta') - (\alpha, \beta)] \in \Sigma = \{(s, t) \in \mathbb{N}^2 \mid 0 \leq t \leq s\}.$$

*Proof.* Suppose otherwise that $\alpha' - \alpha < \beta' - \beta$. Applying Example 2.4(i) to $x^{\alpha}\partial^{\beta} \in Wx^{\alpha}\partial^{\beta}$, we get

$$\{x^{\beta'-\beta+\alpha-1}\partial^{\beta+i} \mid 0 \leq i \leq \beta' - \beta - 1\} \subset Wx^{\alpha}\partial^{\beta}.$$

Since $x^{\alpha'}\partial^{\beta'} \in Wx^{\alpha}\partial^{\beta}$, we have

$$x^{\beta'-\beta+\alpha-1}\partial^{\beta'} = x^{(\beta'-\beta)-(\alpha'-\alpha)-1}(x^{\alpha'}\partial^{\beta'}) \in Wx^{\alpha}\partial^{\beta}.$$

Therefore,

$$\{x^{\beta'-\beta+\alpha-1}\partial^{\beta+i} \mid 0 \leq i \leq \beta' - \beta\} \subset Wx^{\alpha}\partial^{\beta}.$$

By using Example 2.4(ii) repeatedly, we obtain $x^{\alpha-1}\partial^{\beta} \in Wx^{\alpha}\partial^{\beta}$, which is not possible in view of the Bernstein filtration on $W$. □

We remark that the same argument in the proof of Lemma 2.5 generalizes to the $n$th Weyl algebra $W$, namely, if $x^{\alpha'}\partial^{\beta'} \in Wx^{\alpha}\partial^{\beta}$ with $\alpha_i \geq 1$ for some $i \in \{1, \ldots, n\}$, then $[(\alpha'_i, \beta'_i) - (\alpha_i, \beta_i)] \in \Sigma = \{(s, t) \in \mathbb{N}^2 \mid 0 \leq t \leq s\}$.

**Example 2.6.** Using Lemma 2.5, one can readily verify that $\{Wx\partial^{\beta} \mid \beta \in \mathbb{N}\}$ is an infinite antichain of monomial ideals in the first Weyl algebra $W$.

**3. Gröbner bases in the Weyl algebra.** In this section, the ground ring $k$ is a field of characteristic 0. We recall some Gröbner bases theory for the Weyl algebra over $k$ developed in [**SST00,** Section 1.1].

A total order $\prec$ on the set $\mathcal{B}$ of monomials in $W$ is called a term order for $W$ if the following two conditions hold:

(1) $1 = x^0\partial^0$ is the $\prec$-smallest element;
(2) $x^\alpha\partial^\beta \prec x^a\partial^b$ implies $x^{\alpha+s}\partial^{\beta+t} \prec x^{a+s}\partial^{b+t}$ for all $(s,t) \in \mathbb{N}^{2n}$.

The initial monomial $\mathrm{in}_\prec(\delta)$ of an element $\delta \in W$ is the commutative monomial $x^\alpha\xi^\beta \in k[x,\xi]$ such that $x^\alpha\partial^\beta$ is the $\prec$-largest monomial appearing in the canonical form of $\delta$. For a $W$-ideal $I$, its initial ideal is the ideal in $k[x,\xi]$ generated by $\{\mathrm{in}_\prec \delta \mid \delta \in I\}$. A finite set $G$ of $W$ is said to be a Gröbner basis for a $W$-left ideal $I$ with respect to $\prec$ if $I$ is generated by $G$ and the initial ideal $\mathrm{in}_\prec I$ is generated by $\{\mathrm{in}_\prec g \mid g \in G\}$. By [**SST00,** Theorem 1.1.10], every left ideal $I$ of $W$ admits a Gröbner basis $G$ with respect to any given term order $\prec$. Note that not every finite monomial generating set of a monomial ideal forms a Gröbner basis. For example, the initial ideal of $I = Wx + W\partial = W$ is $k[x,\xi]$, which is not generated by $x$ and $\xi$. Nontheless, we have the following analog of the normal form algorithm: every element $\delta \in W$ has a unique normal form $\bar{\delta}^G \in W$ with respect to $G$ such that $\delta \equiv \bar{\delta}^G$ modulo $I$ and that every monomial appearing in the canonical form of $\bar{\delta}^G$ is not divisible by $\Psi(\mathrm{in}_\prec g)$ for any $g \in G$. Here, a monomial $x^\alpha\partial^\beta$ is said to be divisible by $x^a\partial^b$ in $W$ if $\alpha_i \geq a_i$ and $\beta_i \geq b_i$ for all $i$. A monomial of $W$ is called a standard monomial of $I$ with respect to $\prec$ if it is not divisible by $\Psi(\mathrm{in}_\prec g)$ for any $g$ in a Gröbner basis $G$ for $I$.

The following lemma is an immediate consequence of the normal form algorithm.

**Lemma 3.1.** *Let $\prec$ be a term order on $W$ and let $I$ be a left ideal of $W$.*

*(1) The images of the standard monomials of $I$ in $W/I$ form a $k$-basis.*
*(2) The map $\Psi : \mathrm{gr}\, W = k[x,\xi] \to W$ induces an isomorphism between the $k$-vector spaces $\mathrm{gr}\, W/\mathrm{in}_\prec I$ and $W/I$ which sends the standard monomials of $\mathrm{in}_\prec I$ in $\mathrm{gr}\, W$ to the standard monomials of $I$ in $W$.*

*(3) If $I$ is homogeneous with respect to an $A$-grading of $W$, then $\mathrm{in}_{\prec} I$ is homogeneous with respect to the induced $A$-grading on $\mathrm{gr}\, W$ and the map $\Psi$ restricts to an isomorphism between $(\mathrm{gr}\, W/\mathrm{in}_{\prec} I)_a$ and $(W/I)_a$ for each $a \in A$. In particular, the Hilbert functions of $I$ and $\mathrm{in}_{\prec} I$ are identical.*

**4. Main techniques.** For the convenience of the reader, we recall the general framework for the construction of the multigraded Hilbert scheme in **[HS04]**.

Fix a commutative ring $k$ and an arbitrary index set $A$. Consider the pair $(T, F)$ of graded $k$-modules $T = \bigoplus_{a\in A} T_a$ with a collection of operators $F = \bigcup_{a,b\in A} F_{a,b}$ where $F_{a,b} \subseteq \mathrm{Hom}_k(T_a, T_b)$ satisfying $F_{b,c} \circ F_{a,b} \subseteq F_{a,c}$ and $\mathrm{id}_{T_a} \in F_{a,a}$. In fact, $(T, F)$ is a small category of $k$-modules with objects $T_a$ and arrows the elements in $F$.

For a commutative $k$-algebra $R$, we denote by $R \otimes T$ the graded $R$-module $\bigoplus_a (R \otimes T_a)$ with operators $\hat{F}_{a,b} = (1_R \otimes -)(F_{a,b})$. A homogeneous submodule $L = \bigoplus_a L_a \subseteq R \otimes T$ is an $F$-submodule if $\hat{F}_{a,b}(L_a) \subseteq L_b$ for all $a, b \in A$. Fix a function $h : A \to \mathbb{N}$. Let $H_T^h(R)$ be the set of $F$-submodules $L \subseteq R \otimes T$ such that $(R \otimes T_a)/L_a$ is a locally free $R$-module of rank $h(a)$ for each $a \in A$. We have the Hilbert functor $H_T^h : \underline{k\text{-Alg}} \to \underline{\mathrm{Set}}$. For any subset $D$ of $A$, denote by $(T_D, F_D)$ the full subcategory of $(T, F)$ with objects $T_a$ and the set of arrows $F_{D,a,b} = F_{a,b}$ for $a, b \in A$. We have a natural transformation of Hilbert functors $H_T^h \to H_{T_D}^h$ given by restriction of degrees.

**Theorem 4.1.** **[HS04,** Theorem 2.2] *Let $(T, F)$ be a graded $k$-module with operators as above. Suppose there exist homogeneous $k$-submodules $M \subseteq N \subseteq T$ and a subset $F' \subseteq F$ satisfying the following conditions:*

(i) *$N$ is finitely generated $k$-module;*
(ii) *$N$ generates $T$ as an $F'$-module;*
(iii) *for every field $K \in \underline{k\text{-Alg}}$ and every $L \in H_T^h(K)$, $M$ spans $(K \otimes T)/L$;*
(iv) *there is a subset $G \subseteq F'$, generating $F'$ as a category, such that $GM \subseteq N$.*

*Then $H_T^h$ is represented by a quasi-projective scheme over $k$.*

We remark that the statement of Theorem 4.1 is slightly stronger than that of Theorem 2.2 in **[HS04]**. However, the same proof works in this setting. Indeed, in the proof of Theorem 2.2 **[HS04]**, only *Step* 6 concerns the conditions (ii) and (iv) where one needs to produce any element in $T$ using elements in $N$ and operators in $F$. But this does not require the full set of $F$. Any subset $F' \subseteq F$ satisfying conditions (ii) and (iv) will do the job.

Moreover, the hypothesis (iii) implies that $\dim_K(K \otimes T)/L = \sum_{a \in A} h(a)$ is finite, so Theorem 4.1 only works for $h$ having finite support. For the general case, we need the following theorem.

**Theorem 4.2.** **[HS04,** Theorem 2.3] *Let $(T, F)$ be graded $k$-modules with operators and $D \subseteq A$ be such that $H^h_{T_D}$ is represented by a scheme over $k$. Assume that for each degree $a \in A$:*

(v) *there is a finite subset $E \subseteq \bigcup_{b \in D} F_{b,a}$ such that $T_a / \sum_{b \in D} E_{b,a}(T_b)$ is a finitely generated $k$-module;*
(vi) *for every field $K \in \underline{k\text{-Alg}}$ and every $L_D \in H^h_{T_D}(K)$, if $L'$ denotes the $F$-submodule of $K \otimes T$ generated by $L_D$, then $\dim(K \otimes T_a)/L'_a \leq h(a)$.*

*Then the natural transformation $H^h_T \to H^h_{T_D}$ makes $H^h_T$ a subfunctor of $H^h_{T_D}$, represented by a closed subscheme of the Hilbert scheme $H^h_{T_D}$.*

To find a suitable finite set $D$ of degrees satisfying the hypotheses (v) and (vi) in Theorem 4.2, we also need the following fact.

**Proposition 4.3.** **[HS04,** Proposition 3.2] *Let $S$ be an $A$-graded polynomial ring. Given a degree function* $\deg : \mathbb{N}^n \to A$ *and a Hilbert function $h : A \to \mathbb{N}$, there is a finite set of degrees $D \subseteq A$ that satisfies the following two properties:*

(g) *Every monomial ideal of $S$ with Hilbert function $h$ is generated by monomials of degrees in $D$, and*
(h′) *every monomial ideal $I$ of $S$ generated in degrees $D$ satisfies: if $h_I(a) = h(a)$ for all $a \in D$, then $h_I(a) \leq h(a)$ for all $a \in A$.*

Such set $D$ in Proposition 4.3 is called a supportive set of degrees in **[HS04]**. There is also a so-called very supportive set $E$ that is used to write down the defining equations of $H^h_S$ in the positive grading case. Since the $A$-grading on $W$ is never positive, we will not pursue the analogous results on very supportive sets here.

**5. Proof of Theorem 1.1.** Fix any Hilbert function $h : A \to \mathbb{N}$ and let $D$ be a finite subset of degrees in $A$. Our first task is to construct $k$-submodules $M, N$ and a subset $F'_D \subseteq F_D$ satisfying the hypotheses in Theorem 4.1 for the graded $k$-modules $W_D = \bigoplus_{a\in D} W_a$ with the set of operators $F_D$ to be defined later.

Let $F$ be the monoid of operators on $W$ generated by multiplications of monomials in $W$. Note that this is slightly different from the polynomial case due to the non-commutativity of $W$. Denote by $F_{a,b}$ the set of all operators in $F$ that send $W_a$ into $W_b$. Then $F = \bigcup_{a,b\in A} F_{a,b}$. Moreover, we have $F_{b,c} \circ F_{a,b} \subseteq F_{a,c}$ for all $a, b, c \in A$ and $F_{a,a}$ contains the identity map on $W_a$ for all $a \in A$. So $(W, F)$ is a small category of $k$-modules with the components $W_a$ of $W$ as objects and elements of $F$ as arrows. Notice also that for a $k$-algebra $R$, an admissible left ideal in $R \otimes_k W$ is equivalent to a left $F$-submodule $L$ of $R \otimes_k W$ such that $(R \otimes_k W_a)/L_a$ is a locally free $R$-module of rank $h(a)$ for each $a \in A$.

Define $F_D := \bigcup_{a,b\in D} F_{a,b}$, then $(W_D, F_D)$ is a full subcategory of $(W, F)$. Consider for each $k$-algebra $R$ the set $H^h_{W_D}(R)$ of all admissible $F_D$-submodules of $R \otimes_k W_D$ and for each $k$-algebra homomorphism $\phi : R \to S$ the map $H^h_{W_D}(\phi) : H^h_{W_D}(R) \to H^h_{W_D}(S)$. There is a natural transformation of Hilbert functors $H^h_W \to H^h_{W_D}$ given by sending $L \in H^h_W(R)$ to $L_D := \bigoplus_{a\in D} L_a \in H^h_{W_D}(R)$.

For each $a \in A$, let $\mathcal{B}_a$ be the set of monomials (excluding $1 = x^0\partial^0$) with degree $a$. Denoted by $G'_a$ the set of minimal elements in $\mathcal{B}_a$ with respect to the partial ordering $x^\alpha\partial^\beta \le x^{\alpha'}\partial^{\beta'} \Leftrightarrow (\alpha, \beta) \le (\alpha', \beta')$. Recall that $x^{\alpha'}\partial^{\beta'}$ is said to be divisible by $x^\alpha\partial^\beta$ if $(\alpha, \beta) \le (\alpha', \beta')$. By Dickson's lemma, we have $G'_a$ is finite for each $a \in A$. For $a, b \in A$, let $G_{a,b}$ be the set of operators on $W$ consisting of left multiplications by elements in $G'_{b-a}$. Denote by $F'_D$ the monoid (category) generated by $G_D := \bigcup_{a,b\in D} G_{a,b}$. For $a, b \in D$, denote by $F'_{a,b}$ the set of all operators in $F'_D$ that send $W_a$ into $W_b$. The following example shows that the strict inequality $F'_D \subsetneq F_D$ can occur.

**Example 5.1.** Consider the $\mathbb{Z}$-grading on the first Weyl algebra $W = k\langle x, \partial\rangle$ with $\deg(x) = -\deg(\partial) = 1$. Let $D = \{0, 2\} \subseteq \mathbb{Z}$. Then $G_D = \{x\partial, x^2, \partial^2\}$. Observe that the element $\partial x^3 \in F_{0,2} \subseteq F_D$ does not lie in the monoid $F'_D$ generated by elements in $G_D$.

The $A$-grading on $W$ induces an $A$-grading on $\operatorname{gr} W$ by setting $\deg \xi := \deg \partial$. The Hilbert function $h : A \to \mathbb{N}$ can be viewed as Hilbert function for ideals in the polynomial algebra $\operatorname{gr} W$ with this induced $A$-grading. Let $\mathcal{C}_D$ be the set of ideals of $\operatorname{gr} W$ generated by monomials in degrees $D$ with Hilbert functions agreeing with $h$ on $D$. Denote by $M'$ the union over all $I \in \mathcal{C}_D$ of the $\Psi$-images of the standard monomials of $I$ in $(\operatorname{gr} W)_D$, i.e.,

$$M' = \left\{ \Psi(x^\alpha \xi^\beta) \,\middle|\, x^\alpha \xi^\beta \in (\operatorname{gr} W)_D \setminus I, \text{ for some } I \in \mathcal{C}_D \right\}.$$

Since $\mathcal{C}_D$ is finite by **[Mac01]**, the set $M'$ is also finite.

Let $N' = G_D M' \cup \left(\bigcup_{a \in D} G'_a\right)$, $M = kM'$, and $N = kN'$. We check that $(W_D, F'_D, F_D), N, M, G_D$ satisfy the hypotheses of Theorem 4.1 which we rewrite as below.

(i) $N$ is a finitely generated $k$-module;
(ii) $N$ generates $W_D$ as an $F'_D$-module;
(iii) for every field $K \in \underline{k\text{-Alg}}$ and every $L_D \in H^h_{W_D}(K)$, $M$ spans $(K \otimes W_D)/L_D$;
(iv) there is a subset $G_D \subseteq F'_D$, generating $F'_D$ as a category, such that $G_D M \subseteq N$.

The conditions (i) and (iv) hold obviously by our construction. For condition (ii), given $a \in D$ and $x^\alpha \partial^\beta \in W_a$, we want to show that $x^\alpha \partial^\beta$ is generated by $N$ over $F'_D$ by induction on the total degree $|\alpha| + |\beta|$ of $x^\alpha \partial^\beta$. If $x^\alpha \partial^\beta \in G'_a \subseteq N$, the statement is automatically true. For $x^\alpha \partial^\beta \in W_a \setminus G'_a$, there exists $x^{\alpha'} \partial^{\beta'} \in G'_a \subseteq N$ such that $x^\alpha \partial^\beta$ is divisible by $x^{\alpha'} \partial^{\beta'}$. Notice that the total degree of the element $(x^\alpha \partial^\beta - x^{\alpha-\alpha'} \partial^{\beta-\beta'} \cdot x^{\alpha'} \partial^{\beta'}) \in W_a$ is strictly less than that of $x^\alpha \partial^\beta$, so by inductive hypothesis it is generated by $N$ over $F'_D$. Therefore, it suffices to show that $x^{\alpha-\alpha'} \partial^{\beta-\beta'} \in W_0$ acts on $W_a$ as a sum of operators in $F'_D$. In fact, we will show every element $x^{\bar\alpha} \partial^{\bar\beta} \in W_0$ acts as a sum of operators in $F'_D$ by induction on the total degree of $x^{\bar\alpha} \partial^{\bar\beta}$. Recall that $F'_D$ is the monoid generated by $\bigcup_{a,b \in D} G'_{a,b}$. In particular, if $x^{\bar\alpha}\partial^{\bar\beta} \in G'_0$, it acts as an operator in $G_{a,a} \subseteq F'_D$ for any $a \in D$. For $x^{\bar\alpha}\partial^{\bar\beta} \in W_0 \setminus G'_0$, there exists $x^{\bar\alpha'}\partial^{\bar\beta'} \in G'_0$ such that $x^{\bar\alpha}\partial^{\bar\beta}$ is divisible by $x^{\bar\alpha'}\partial^{\bar\beta'}$. Since $x^{\bar\alpha-\bar\alpha'}\partial^{\bar\beta-\bar\beta'}$ and $(x^{\bar\alpha}\partial^{\bar\beta} - x^{\bar\alpha-\bar\alpha'}\partial^{\bar\beta-\bar\beta'} \cdot x^{\bar\alpha'}\partial^{\bar\beta'})$ are in $W_0$ and have total degree strictly less than $x^{\bar\alpha}\partial^{\bar\beta}$, they both act as a sum of operators in $F'_D$. We conclude that

$$x^{\bar\alpha}\partial^{\bar\beta} = (x^{\bar\alpha}\partial^{\bar\beta} - x^{\bar\alpha-\bar\alpha'}\partial^{\bar\beta-\bar\beta'} \cdot x^{\bar\alpha'}\partial^{\bar\beta'}) + x^{\bar\alpha-\bar\alpha'}\partial^{\bar\beta-\bar\beta'} \cdot x^{\bar\alpha'}\partial^{\bar\beta'}$$

also acts as a sum of operators in $F'_D$. This establishes condition (ii). We need the following lemma to verify condition (iii).

**Lemma 5.2.** *Let $R \in \underline{k\text{-Alg}}$, $L_D \in H^h_{W_D}(R)$, and $L \subseteq R \otimes_k W$ be the left ideal generated by $L_D$. Then $L_a = L_{D,a}$ for all $a \in D$.*

*Proof.* Observe that for $a \in D$,

$$L_a = \sum_{b\in D} F_{b,a}(L_{D,b}) \supseteq F_{a,a}(L_{D,a}) \supseteq L_{D,a}.$$

Conversely, we have $F_{b,a}(L_{D,b}) \subseteq L_{D,a}$ for any $a, b \in D$, since $L_D \in H^h_{W_D}(R)$ is an $F_D$-submodule of $R \otimes_k W_D$. □

For condition (iii), fix a field $K \in \underline{k\text{-Alg}}$ and an $F_D$-submodule $L_D \in H^h_{W_D}(K)$. Let $L \subseteq K \otimes_k W$ be the ideal generated by $L_D$. Fix any term order $\prec$ on $W$ and let $I = \operatorname{in}_\prec L \subset \operatorname{gr} W$ be the initial ideal of $L$ with respect to $\prec$. By Lemma 3.1(3), the Hilbert functions $h_I$ and $h_L$ coincide. Hence $h_I(a) = h_{L_D}(a)$ for all $a \in D$ by Lemma 5.2. In particular, the ideal $I \in \mathcal{C}_D$ and by Lemma 3.1(2) $M'$ spans $(K \otimes_k W_D)/L_D$, which is exactly the statement of condition (iii).

At this point, we have shown by using Theorem 4.1 that $H^h_{W_D}$ is represented by a quasi-projective scheme for any finite set $D$ of degrees in $A$. To complete the proof of Theorem 1.1, it remains to verify the following conditions (v) and (vi) (for each degree $a \in A$) in Theorem 4.2 for some suitable finite subset $D$ of $A$.

- (v) there is a finite subset $E \subseteq \bigcup_{b\in D} F_{b,a}$ such that $W_a / \sum_{b\in D} E_{b,a}(W_b)$ is a finitely generated $k$-module;
- (vi) for every field $K \in \underline{k\text{-Alg}}$ and every $L_D \in H^h_{W_D}(K)$, if $L$ denotes the $F$-submodule of $K \otimes W$ generated by $L_D$, then $\dim(K \otimes W_a)/L_a \leq h(a)$.

Applying Proposition 4.3 to the case where $S = \operatorname{gr} W$ with the induced degree function $\deg : \mathbb{N}^{2n} \to A$ given by $\deg(\xi) = \deg(\partial)$, we can find a finite subset $D$ of $A$ that satisfies the conditions (g) and (h′) for $\operatorname{gr} W$ with respect to the same Hilbert function $h$. From now, fix such a finite set $D$. The goal is to conclude from Theorem 4.2 that the natural transformation $H^h_W \to H^h_{W_D}$ makes $H^h_W$ a subfunctor of $H^h_{W_D}$, represented by a closed subscheme of the Hilbert scheme $H^h_{W_D}$. We may assume there exists an admissible $L$ of $W$ whose Hilbert function $h_L = h$, for otherwise the statement of Theorem 4.2 is empty. Choose any term order $\prec$ on $W$. By Lemma 3.1(3), the Hilbert function $h_{\mathrm{in}_\prec L}$ of the initial ideal $\mathrm{in}_\prec L$ of $L$ in $\operatorname{gr} W$ coincides with $h$, and hence $\mathrm{in}_\prec L$ is generated in degrees $D$ by the condition (g) in Proposition 4.3. Since the s-pair of two homogeneous elements in the Weyl algebra is still homogeneous, there exists a Gröbner basis for $L$ consisting of homogeneous elements in degrees $D$. In particular, the ideal $L$ of $W$ is also generated in degrees $D$. Therefore, for each $a \in A$ the component $L_a = \sum_{b \in D} F_{b,a}(L_b)$ and it has finite $k$-codimension $h(a)$ in $W_a$.

To verify condition (v), it suffices to find a finite subset $E \subseteq \bigcup_{b \in D} F_{b,a}$ such that for any $b \in D$, $a \in A$,

$$F_{b,a}(L_b) \subseteq \sum_{b' \in D} E_{b',a}(W_b).$$

Take $E_{b,a} = G_{b,a}$ and let $E = \bigcup_{b \in D} E_{b,a}$. We claim that in fact $F_{b,a}(W_b) \subseteq E_{b,a}(W_b) = G_{b,a}(W_b)$. Since each operator in $F_{b,a}$ (which is a product of monomials) can be written as a sum of monomials in degree $a - b$ by Corollary 2.2, we check only that if $\deg(x^\alpha \partial^\beta) = a - b$ then $x^\alpha \partial^\beta (W_b) \subseteq G_{b,a}(W_b)$. It is certainly true that $x^\alpha \partial^\beta (W_b) \subseteq G_{b,a}(W_b)$ when $x^\alpha \partial^\beta \in G'_{a-b}$. In general, we have $x^\alpha \partial^\beta$ is divisible by some element $x^{\alpha'} \partial^{\beta'} \in G'_{a-b}$ and by inductive hypothesis

$$[x^\alpha \partial^\beta (W_b) - x^{\alpha'} \partial^{\beta'} \cdot x^{\alpha - \alpha'} \partial^{\beta - \beta'} (W_b)] \subseteq G_{b,a}(W_b).$$

Since $\deg(x^{\alpha-\alpha'} \partial^{\beta-\beta'}) = 0$, we have $x^{\alpha-\alpha'} \partial^{\beta-\beta'} (W_b) \subseteq W_b$ and hence $x^\alpha \partial^\beta (W_b) \subseteq G_{b,a}(W_b)$ as desired.

For condition (vi), fix a field $K \in \underline{k}\text{-Alg}$, an element $L_D \in H^h_{W_D}(K)$, and let $L \subseteq K \otimes_k W$ be the ideal generated by $L_D$. By Lemma 5.2, $L_a = L_{D,a}$ and hence $h_L(a) = h(a)$ for all $a \in D$. Also, we have $h_L(a) = h_{\mathrm{in}_\prec L}(a)$ for all $a \in A$ by Lemma 3.1(3). Let $I$ be the monomial ideal in $\operatorname{gr} W$ generated by $(\mathrm{in}_\prec L)_D$. Then $I_a = (\mathrm{in}_\prec L)_a$ for all $a \in D$ by the same argument of Lemma 5.2, and hence $h_I(a) = h_{\mathrm{in}_\prec L}(a) = h_L(a) = h(a)$ for all $a \in D$. Therefore, by condition (h′) of Proposition 4.3, $h_L(a) = h_{\mathrm{in}_\prec L}(a) \leq h_I(a) \leq h(a)$ for all $a \in A$. This establishes condition (vi).

**Example 5.3.** Let $k$ be a field. Consider the finest possible $A$-grading on $W$ where $A = \mathbb{Z}^n$ and $\deg(x_i) = -\deg(\partial_i) = e_i$. Under this $A$-grading, any homogeneous ideals are generated by elements in $W$ of the form $x^a p(\theta)\partial^b$, $a, b \in \mathbb{N}^n$. By [**SST00,** Lemma 2.3.1], such ideals are the torus-fixed ideals of $W$ which are used in the algorithms for solving systems of linear partial differential equations.

Fixing a Hilbert function $h : A \to \mathbb{N}$, we remark that if $I, J \in H_W^h(k)$ and if $I$ is holonomic, then $J$ is also holonomic. Indeed, using the notations in [**SST00**], the Hilbert functions of $\mathrm{in}_{\prec_{(0,e)}} I$ and $\mathrm{in}_{\prec_{(0,e)}} J$ coincide by Lemma 3.1. Therefore, by [**SST00,** Theorem 1.1.6] the ideals $\mathrm{in}_{(0,e)} I$ and $\mathrm{in}_{(0,e)} J$ in $\operatorname{gr} W$ also have the same Hilbert functions under the $A$-grading inherited from that of $W$. In particular, $\dim \mathrm{in}_{(0,e)} I = \dim \mathrm{in}_{(0,e)} J$ and the holonomicity of $J$ follows.

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