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The strong monodromy conjecture for monomial ideals on toric varieties

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Dedicated to Professor Gennady Lyubeznik on the occasion of his 60th birthday.

ABSTRACT
We compute Denef and Loeser's motivic zeta function associated to a monomial ideal on an affine toric variety, generalizing a result of Howald, Mustaţă, and Yuen. We also investigate the relation between the poles of the motivic zeta function and the roots of their corresponding Bernstein–Sato polynomial defined by the first author and Matusevich.

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1. Introduction

The strong monodromy conjecture of Igusa states that the real part of a pole of the p-adic zeta function associated to a hypersurface defined by a polynomial f is a root of the Bernstein–Sato polynomial of f [12]. A fascinating consequence of this conjecture is a link between the poles of Igusa zeta functions and the monodromy eigenvalues of Milnor fibers of f. As a generalization of Igusa zeta function, Denef and Loeser introduce their motivic zeta function using motivic integration [6, 7]. We will focus on the special type of motivic zeta function $Z(X, \mathcal{I};s)$ associated to an ideal sheaf $\mathcal{I}$ on an algebraic variety X, whose definition will be recalled in Section 2. For more information about this subject, we refer to the surveys [5, 17, 14].

On the other hand, Budur, Mustaţă, and Saito define a generalization of Bernstein–Sato polynomial $b_I^X(\mathfrak{s})$ for an ideal I on a smooth affine variety X [2]. Their definition is adopted in [11] to extend the notion of Bernstein–Sato polynomial $b_I^{U_\sigma}(\mathfrak{s})$ for an ideal I on an affine (normal) toric variety U_σ. It is then curious for us to see whether the analogous strong Monodromy Conjecture holds in the toric setting. In [10], this conjecture under the p-adic setting is verified for the case where I is a monomial ideal on an affine space $\mathbb{A}^n$. The key ingredient in their proof is an explicit formula of the p-adic zeta function associated to the pair $(\mathbb{A}^n, I)$. It turns out that the real parts of candidate poles of the p-adic zeta function associated to $(\mathbb{A}^n, I)$ come from the rays of the normal fan Δ_I of I. So one can test the conjecture using the combinatorial description in [3] of the roots of $b_I^{\mathbb{A}^n}(\mathfrak{s})$ in terms of the Newton polyhedron of I.

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The purpose of this paper is to test the conjecture for the motivic zeta function $Z(U_\sigma, I;s)$ and the Bernstein–Sato polynomial $b_I^{U_\sigma}(\mathfrak{s})$ assocaited to a monomial ideal I on an affine toric variety U_σ. To do this, we need an explicit formula for $Z(U_\sigma,I;s)$. This relies on a change of variables formula of motivic zeta functions obtained in [7] applying to a toric log resolution $h: X(\Delta)\to U_\sigma$ of (U_σ, I) and the description of the contact locus of a monomial ideal on a toric variety in [13] or Section 3. As a result, the motivic zeta function $Z(U_\sigma, I;s)$ has a similar description as in the case studied in [10], except that a pole of $Z(U_\sigma, I;s)$ may come from a ray of the fan Δ that is not a ray of the normal fan Δ_I. In fact, we show in Example 6.2 and 6.3 that such a ray does possibly produce a pole of $Z(U_\sigma, I;s)$ that is not a root of the b-function $b_I^{U_\sigma}(\mathfrak{s})$. However, under a mild assumption we can still prove in Theorem 6.4 that the poles obtained from the rays of the normal fan Δ_I are roots of $b_I^{U_\sigma}(\mathfrak{s})$. See Section 2.4 for the definition of a pole of the motivic zeta function $Z(X, \mathcal{I}; s)$ as well as the analogous strong Monodromy Conjecture in the motivic setting (Conjecture 2.1).

Our explicit computation of the toric motivic zeta functions is one of the first few examples where the ambient space is singular. We mention the work of Veys on the motivic zeta function of an effective divisor E on a $\mathbb{Q}$-Gorenstein variety X [15] with the condition that the singular locus of X is contained in the support of E. See also [16] for a study of relevant zeta functions on varieties with Kawamata log terminal singularities.

2. Motivic Igusa zeta functions of Denef and Loeser

We state some definitions and results from [7].

2.1.

Let k be a field of characteristic 0 and let $\mathcal{M}$ be the Grothendieck ring of algebraic varieties over k. Set $\mathbb{L} := [\mathbb{A}_k^1] \in \mathcal{M}$ and denote $\hat{\mathcal{M}}$ the completion of $\mathcal{M}[\mathbb{L}^{-1}]$ with respect to the filtration $\{\mathcal{F}^{\mathrm{m}}\mathcal{M}[\mathbb{L}^{-1}]\}_{\mathrm{m}\in\mathbb{Z}}$, where $\mathcal{F}^{\mathrm{m}}\mathcal{M}[\mathbb{L}^{-1}]$ is the subgroup of $\mathcal{M}[\mathbb{L}^{-1}]$ generated by $\{[S]\mathbb{L}^{-i}|i-\dim S\geq m\}$.

For an algebraic variety X of dimension d over k, denote $\mathcal{L}(X)$ and $\mathcal{L}_m(X)$ the arc space and the m-th jet scheme of X respectively. Let $\pi_m : \mathcal{L}(X) \to \mathcal{L}_m(X)$ be the canonical morphism corresponding to truncation of arcs. For a semi-algebraic subset A of $\mathcal{L}(X)$, Denef and Loeser define the motivic measure $\mu(A) \in \widehat{\mathcal{M}}$ of A. We will not recall the definition of a semi-algebraic of $\mathcal{L}(X)$ nor the construction of the motivic measure μ. Instead, we only mention some properties that will be used later.

1. If a semi-algebraic set A of $\mathcal{L}(X)$ is contained in $\mathcal{L}(S)$ for some closed subvariety S of X with $\dim S<\dim X$, then $\mu(A) = 0$.
2. If A_i, $i\in\mathbb{N}$, are semi-algebraic and mutually disjoint, then $A := \cup_{,i\in\mathbb{N}}A_i$ is semi-algebraic and $\sum_{i\in\mathbb{N}}\mu(A_i)$ converges to $\mu(A)$ in $\widehat{\mathcal{M}}$.
3. If $B\subset\mathcal{L}_m(X)$ is constructible then $\pi_m^{-1}(B)$ is semi-algebraic. Such semi-algebraic set $\pi_m^{-1}(B)$ will be called a cylinder or a constructible set in $\mathcal{L}(X)$. Moreover, if X is smooth and $A = \pi_m^{-1}(B)$ is constructible, then

$$\mu(A) = \frac{[\pi_m(A)]}{\mathbb{L}^{(m+1)d}} \in \widehat{\mathcal{M}}$$

which does not depend on m.

2.2.

Let $\mathcal{I}$ be a coherent ideal sheaf on X, and consider the function $ord_t\mathcal{I}:\mathcal{L}(X)\to\mathbb{N}\cup\{\infty\}$ given by $\gamma\mapsto \min_g \mathrm{ord}_t g(\tilde{\gamma})$, where the minimum is taken over all g in the stalk $\mathcal{I}_{\pi_0(\gamma)}$ of $\mathcal{I}$ at $\pi_0(\gamma)$ and $\tilde{\gamma} \in$

$\mathcal{L}(X)(k_\gamma) = X(k_\gamma[[t]])$ is the corresponding rational point of $\gamma\in\mathcal{L}(X)$ over the residue field k_γ of γ on $\mathcal{L}(X)$. The motivic Igusa zeta function associated the ideal $\mathcal{I}$ on X is defined as

$$Z(X,\mathcal{I};s) = \int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_t\mathcal{I}\cdot s}\, d\mu := \sum_{n\geq 0} \mu(\mathrm{Cont}^n(\mathcal{I}))\cdot \mathbb{L}^{-ns} \in \widehat{\mathcal{M}}[[\mathbb{L}^{-s}]] \tag{2.1}$$

where the fiber $\mathrm{Cont}^n(\mathcal{I}) := \{\gamma \in \mathcal{L}(X) \mid \mathrm{ord}_t\mathcal{I}(\gamma) = n\} = (\mathrm{ord}_t\mathcal{I})^{-1}(n)$ of the function $\mathrm{ord}_t\mathcal{I}$ is the n-th contact locus of $\mathcal{I}$.

2.3.

For a log resolution $h{:}Y\to X$ of the pair $(X,\mathcal{I})$, let $\mathcal{L}(h){:}\mathcal{L}(Y)\to\mathcal{L}(X)$ be the induced map on the corresponding arc spaces. There is a change of variables formula

$$\begin{aligned}\int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_t\mathcal{I}\cdot s}\, d\mu &= \int_{\mathcal{L}(Y)} \mathbb{L}^{-(\mathrm{ord}_t\mathcal{I}\circ\mathcal{L}(h)\cdot s+\mathrm{ord}_t\mathrm{Jac}_h)}\, d\mu \\ &:= \sum_{i,j\in\mathbb{N}} \mu(\mathrm{Cont}^i(\mathcal{I}\cdot\mathcal{O}_Y)\cap\mathrm{Cont}^j(\mathrm{Jac}_h))\cdot\mathbb{L}^{-(is+j)}.\end{aligned} \tag{2.2}$$

Here, the Jacobian ideal sheaf Jac_h of h is by definition the ideal sheaf of $\mathcal{O}_Y$ such that the image of the morphism $h^*\Omega^d_X\to\Omega^d_Y$ induced by pulling-back d-forms can be written as $\mathrm{Jac}_h\cdot\Omega^d_Y$. Note that Ω^d_X is the dth wedge product of the sheaf of Kähler differentials Ω_X on X. For more details, except for [7], the reader is also referred to [8] in which another proof of this formula is given.

2.4.

Now, let I be an ideal on an affine variety X. Assume X is either smooth or a normal toric variety. Then we have the notion of b-function $b^X_I(\mathfrak{s})$ for the pair (X,I) [2, 11]. The strong Monodromy Conjecture in the motivic setting can be formulated *as* the following.

Conjecture 2.1. *There exists a finite subset S of $\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ such that*

$$Z(X,\mathcal{I};s) \in \widehat{\mathcal{M}}\left[\mathbb{L}^{-s}, \frac{1}{1-\mathbb{L}^{-as-b}}\right]_{(a,b)\in S}$$

and such that $-\frac{b}{a}$ is a root of $b^X_I(\mathfrak{s})$.

Under this conjecture, we can express the motivic zeta function $Z(X,\mathcal{I};s)$ as a rational function $\frac{N(\mathbb{L}^{-s})}{D(\mathbb{L}^{-s})}$ for $N(\mathbb{L}^{-s}), D(\mathbb{L}^{-s}) \in \mathcal{M}[\hat{\mathbb{L}}^{-s}]$. A rational number $-\frac{b}{a}$ will be called a pole of the motivic zeta function $Z(X,\mathcal{I};s)$ if $1-\mathbb{L}^{-as-b}$ divides $D(\mathbb{L}^{-s})$ but not divides $N(\mathbb{L}^{-s})$.

3. Arc spaces of toric varieties

We recall some results in [13] concerning the arc spaces of toric varieties. Throughout the paper, we will use freely the results in toric geometry as described in [9].

3.1.

Let M be the free abelian group $\mathbb{Z}^d$ and N its dual $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ together with the canonical pairing $\langle,\rangle{:}N\times M\to\mathbb{Z}$. We denote $M\otimes_{\mathbb{Z}}\mathbb{R}$ and $N\otimes_{\mathbb{Z}}\mathbb{R}$ by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, respectively. The pair $\langle,\rangle$ naturally extends to a pairing $N_{\mathbb{R}}\times M_{\mathbb{R}}\to\mathbb{R}$ which we also denote by $\langle,\rangle$. For a linear subspace $W\subset N_{\mathbb{R}}$, the induced pairing $(N_{\mathbb{R}}/W)\times W^{\perp}\to\mathbb{R}$ is also denoted by $\langle,\rangle$. Let $\rho : N_{\mathbb{R}}\to N_{\mathbb{R}}/W$ be the projection. We have $\langle v,u\rangle = \langle\rho(v), u\rangle$ for $v\in N_{\mathbb{R}}$ and $u\in W^{\perp}$.

Let $X = X(\Delta)$ be the toric variety over a field k corresponding to a d-dimensional fan Δ in $N_{\mathbb{R}}$ and denote by $T: = Spec\ k[M]$ the dense open torus in X. For a cone $\tau \in \Delta$, denote $X(\tau) = \overline{\text{orb}\ \tau}$ the orbit closure corresponding to τ. The toric variety $X(\tau)$ corresponds to the the fan Δ_τ in $\mathbb{N}_{\mathbb{R}}/\mathbb{R}\tau$ that consists of the image $\overline{\sigma}$ in $\mathbb{N}_{\mathbb{R}}/\mathbb{R}\tau$ of the cones $\sigma \in \Delta$, $\tau < \sigma$. The affine toric subvariety $U_{\overline{\sigma}} \subset X(\tau)$ is Spec $k[\tau^\perp \cap \sigma^\vee \cap M]$. Denote by N_τ the image of N in $N_{\mathbb{R}}/\mathbb{R}\tau$.

3.2.

For a field extension $K \supseteq k$, let η be the generic point of $Spec\ K[[t]]$. The arc space $\mathcal{L}(X)$ naturally decomposes as $\mathcal{L}(X) = \sqcup_{\tau\in\Delta}\mathcal{L}(X)(\tau)$ where

$$\mathcal{L}(X)(\tau) = \{\gamma \in \mathcal{L}(X) \mid \gamma(\eta) \in \text{orb}\ \tau\}$$

whose closure in $\mathcal{L}(X)$ is $\overline{\mathcal{L}(X)(\tau)} = \mathcal{L}(X(\tau))$. This decomposition can be further refined to $\mathcal{L}(T)$-orbits of $\mathcal{L}(X)$ under the action of $\mathcal{L}(T)$. In fact, for each $\tau\in\Delta$, there is a one-to-one correspondence

$$\{\mathcal{L}(T)\cdot\gamma \mid \gamma \in \mathcal{L}(X)(\tau)\} \leftrightarrow |\Delta_\tau| \cap N_\tau$$

that sends the $\mathcal{L}(T)$-orbit of $\gamma \in \mathcal{L}(X)(\tau)$ to a lattice point $v \in |\Delta_\tau| \cap N_\tau$ satisfying, for $u\in M$, $\text{ord}_t\gamma^*(x^u) = \langle v, u\rangle$. Under this correspondence, we write $\mathcal{L}(T)_v$ for the $\mathcal{L}(T)$-orbit $\mathcal{L}(T)\cdot\gamma$. Note that the orbits $\mathcal{L}(T)_v$ are cylinders in $\mathcal{L}(X)$. The motivic measure

$$\mu(\mathcal{L}(T)_v) = 0, \ \text{ for } 0 \gneqq \tau \text{ and } v \in |\Delta_\tau| \cap N_\tau, \tag{3.1}$$

since $\mathcal{L}(T)_v \subseteq \mathcal{L}(X(\tau))$ and since $X(\tau)$ is a closed subvariety of X with $\dim X(\tau) < \dim X$.

3.3.

For a torus invariant ideal sheaf $\mathcal{I}$ on X, the n-th contact locus $Cont^n(\mathcal{I})$ is a disjoint union of $\mathcal{L}(T)$-orbits. In fact, it follows from (3.1) above and from Theorem 4.1 and Lemma 5.9 in [13] that we have the following proposition.

Proposition 3.1 ([13]). *An orbit $\mathcal{L}(T)\cdot\gamma = \mathcal{L}(T)_v$, $v \in |\Delta_\tau| \cap N_\tau$, is contained in* $\text{Cont}^n(\mathcal{I})$ *if and only if, for any cone $\overline{\sigma} \in \Delta_\tau$ containing v, we have $\mathcal{L}(T)_v \subseteq \mathcal{L}(U_{\overline{\sigma}})$ and*

$$n = \min_{x^u\in\mathcal{I}(U_{\overline{\sigma}})} \text{ord}_t\gamma^*(x^u) = \min_{x^u\in\mathcal{I}(U_{\overline{\sigma}})} \langle v, u\rangle.$$

In particular, the motivic measure $\mu(\text{Cont}^n(\mathcal{I})) = \sum_v \mu(\mathcal{L}(T)_v)$ where the sum runs through all $v\in|\Delta|\cap N$ satisfying $n = \min_{x^u\in\mathcal{I}(U_\sigma)}\langle v, u\rangle$ for all $\sigma\in\Delta$ containing v.

4. Bernstein–Sato polynomials on toric varieties

4.1.

Let Σ be the cone $\mathbb{R}^n_{\geq 0}$ in $\mathbb{R}^n$ and $U_\Sigma = \mathbb{A}^n_k$ be the corresponding affine space over a field k of characteristic 0. Denote by D the n-th Weyl algebra. In [2], Budur, Mustaţă, and Saito define a generalization of Bernstein–Sato polynomials $b_J^{U_\Sigma}(\mathfrak{s})$ for an ideal $J = \langle f_1, \dots, f_r\rangle$ on U_Σ as follows.

Let $s_1, \dots, s_r$ be indeterminates over k, consider

$$k[x_1, \dots, x_n]\left[\prod_{i=1}^r f_i^{-1}, s_1, \dots, s_r\right]\prod_{i=1}^r f_i^{s_i}.$$

This is a $D[s_{ij}]$-module, where $s_{ij} = s_i t_i^{-1} t_j$, and the action of the operator t_i is given by $t_i(s_j) = s_j + \delta_{ij}$ (the Kronecker delta). The *Bernstein–Sato polynomial* or *b-function* associated to $J =$

$\langle f_1, ..., f_r \rangle$ is defined to be the monic polynomial $b_J^{U_\Sigma}(\mathfrak{s})$ of the lowest degree in $\mathfrak{s} = \sum_{i=1}^r s_i$ satisfying a relation of the form

$$b_J^{U_\Sigma}(\mathfrak{s}) \prod_{i=1}^{r} f_i^{s_i} = \sum_{k=1}^{r} P_k t_k \prod_{i=1}^{r} f_i^{s_i}, \tag{4.1}$$

where $P_1, ..., P_r \in D[s_{ij} \mid i, j \in \{1, ..., r\}]$. It is shown in [2] that $b_J^{U_\Sigma}(\mathfrak{s})$ is a nonzero polynomial and is independent of the generating set of J.

4.2.

When J is a monomial ideal on an affine space U_Σ, the roots of $b_J^{U_\Sigma}(\mathfrak{s})$ can be completely described in terms of the Newton polyhedron P_J of the ideal J [3].

Denote $\Gamma := \Sigma^\vee \cap \mathbb{Z}^n$ and $\Gamma_I := \{u \in \Gamma \mid x^u \in I\}$. Let Q be a face of P_J. Denote by V_Q the vector subspace of $\mathbb{R}^d$ generated by Q. Let M_Q the subset of $\mathbb{Z}^n$ such that $M_Q - \mathbb{1}$ is the semi-subgroup of $\mathbb{Z}^n$ generated by $\{a - b \mid a \in \Gamma_J$ and $b \in \Gamma_J \cap Q\}$, where $\mathbb{1} = (1, \ldots, 1) \in \mathbb{Z}^n$. Let $M'_Q := M_Q + b_0$ be the shift of M_Q by any lattice point $b_0 \in \Gamma_J \cap Q$, which does not depend on the choice of b_0.

Theorem 4.1 ([3]). *The set of roots of $b_J^{U_\Sigma}(\mathfrak{s})$ is the union of the sets*

$$R_Q := \{-L_Q(a) \mid a \in (M_Q \setminus M'_Q) \cap V_Q\}$$

for faces Q of P_J that is not contained in any coordinate hyperplane, where $L_Q : \mathbb{R}^n \to \mathbb{R}$ is a linear functional satisfying $L_Q(a) = 1$ for all $a \in Q$.

4.3.

Now, let U_σ be the affine toric variety corresponding a d-dimensional cone σ in $\mathbb{R}^d$ and let I be an ideal on U_σ. Apply the same functional equation (4.1) using the differential operators on U_σ, one can define the b-function $b_I^{U_\sigma}(\mathfrak{s})$ for the pair (U_σ, I). In [11], the b-function $b_I^{U_\sigma}(\mathfrak{s})$ is related to the b-function $b_J^{U_\Sigma}(\mathfrak{s})$ of certain ideal J on an affine space U_Σ. We recall the statement as follows.

Let $\{v_1, ..., v_n\}$ be the set of primitive generators of the rays of σ, each of which corresponds to a facet of $\sigma^\vee$. Consider the linear map $F : \mathbb{R}^d \to \mathbb{R}^n$ that sends $u \in \mathbb{R}^d$ to $(\langle v_1, u \rangle, ..., \langle v_n, u \rangle) \in \mathbb{R}^n$. The map F induces a ring homomorphism $k[\sigma^\vee \cap \mathbb{Z}^d] \to k[\Sigma^\vee \cap \mathbb{Z}^n]$ by sending $f := \sum_j \lambda_j x^{u_j}$ to $F(f) := \sum_j \lambda_j y^{F(u_j)}$, where $x^a := \prod_i x_i^{a_i}$ and $y^b := \prod_j y_j^{b_j}$ for $a \in \sigma^\vee \cap \mathbb{Z}^d$ and $b \in \Sigma^\vee \cap \mathbb{Z}^n$. For an ideal $I = \langle f_1, ..., f_r \rangle$ of $k[\sigma^\vee \cap \mathbb{Z}^d]$, consider the ideal $J = \langle F(f_1), ..., F(f_r) \rangle$ in $k[\Sigma^\vee \cap \mathbb{Z}^n]$.

Theorem 4.2 ([11]). $b_I^{U_\sigma}(\mathfrak{s}) = b_J^{U_\Sigma}(\mathfrak{s})$

We mention that Theorem 4.2 is used in [11] to relate the b-function $b_I^{U_\sigma}(\mathfrak{s})$ with multiplier ideals of the pair (U_σ, I).

5. The case of monomial ideals on affine spaces

Let $X = U_\tau$ be the d-dimensional affine space over a field k of characteristic 0 where $\tau \subset N_\mathbb{R}$ is a pointed smooth cone. Let J be a nonzero proper monomial ideal on X.

5.1.

By Proposition 3.1, the motivic zeta function in (2.1) can be express as

$$Z(U_\tau, J; s) = \sum_{n\geq 0} \mu(\mathrm{Cont}^n(J)) \cdot \mathbb{L}^{-ns} = \sum_{v\in\tau\cap N} \mu(\mathcal{L}(T)_v) \cdot \mathbb{L}^{-\nu_J(v)s}$$

where $\nu_J(v) := \min_{x^u\in J}\langle v, u\rangle$.

5.2.

Let $e_1, ..., e_d$ be the primitive generators on the rays of $\tau^\vee$. Then $x^{e_1}, ..., x^{e_d}$ generate the polynomial algebra $k[\tau^\vee \cap M]$. For $m \geq \max_{1\leq i\leq d}\langle v, e_i\rangle$, consider the constructible set $C_m \subset \mathcal{L}_m(X)$ consisting of m-jets γ with $\mathrm{ord}_t\gamma^*(x^{e_i}) = \langle v, e_i\rangle$. Then $\pi_m^{-1}(C_m) = \mathcal{L}(T)_v$ is a cylinder and

$$[\pi_m(\mathcal{L}(T)_v)] = [C_m] = (\mathbb{L}-1)^d \cdot \mathbb{L}^{md} \cdot \mathbb{L}^{-\langle v, \sum_i e_i\rangle} \in \mathcal{M}.$$

Putting $e_\tau = \sum_i e_i$, the motivic measure of $\mathcal{L}(T)_v$ can be expressed as

$$\mu(\mathcal{L}(T)_v) = \frac{[C_m]}{\mathbb{L}^{m+1}d} = (1-\mathbb{L}^{-1})^d \cdot \mathbb{L}^{-\langle v, e_\tau\rangle}. \tag{5.1}$$

Therefore, the motivic zeta function

$$Z(U_\tau, J; s) = \sum_{v\in\tau\cap N} (1-\mathbb{L}^{-1})^d \cdot (\mathbb{L}^{-1})^{\nu_J(v)s+\langle v, e_\tau\rangle}. \tag{5.2}$$

5.3.

Let Δ_J be the normal fan of the Newton polyhedron P_J of J. The fan Δ_J consists of cones

$$\sigma_Q := \{v \in N_{\mathbb{R}} | \langle v, u\rangle \leq \langle v, u'\rangle \text{ for u } \in Q \text{ and } u' \in P_J\}$$

corresponding to faces Q of P_J. Note that if w is a vertex of P_J such that $v\in\tau\cap N$ is contained in σ_w, then $\nu_J(v) = \langle v,w\rangle$.

The following theorem collects the main results in [10]. Although it was originally stated for the cone $\Sigma = \mathbb{R}^n_{\geq 0}$, we rewrite it for the smooth cone τ by using the natural bijection $\tau \cong \Sigma$.

Theorem 5.1 ([10]). *The motivic zeta function $Z(U_\tau,J;s)$ is a rational function of $\mathbb{L}^{-s}$. Each pole of $Z(U_\tau,J;s)$ can be written as the form $-\frac{\langle v,e_\tau\rangle}{\nu_J(v)}$ for a primitive generator v of some ray of the normal fan Δ_J of J. Moreover, any such number $-\frac{\langle v,e_\tau\rangle}{\nu_J(v)}$ is a root of the Bernstein–Sato polynomial $b_J^{U_\tau}(\mathfrak{s})$. In particular, Conjecture 2.1 holds for monomial ideals on affine spaces.*

6. The case of monomial ideals on affine toric varieties

Let $X = U_\sigma$ be an affine toric variety of dimension d that corresponds to some cone σ in $N_{\mathbb{R}}$. Let I be a nonzero proper monomial ideal on X. Consider a toric log resolution $h{:}Y = X(\Delta)\to X$ of the pair (X,I). The toric map h factors through the normalized blowing-up $X(\Delta_I)$ of X along I. By the change of variables formula (2.2), the motivic zeta function (2.1) can be written as

$$Z(U_\sigma, I; s) := \sum_{i,j\in\mathbb{N}} \mu(\mathrm{Cont}^i(I\cdot\mathcal{O}_Y) \cap \mathrm{Cont}^j(\mathrm{Jac}_h)) \cdot \mathbb{L}^{-(is+j)}. \tag{6.1}$$

Let $v \in \sigma \cap N$ and let $\tau \in \Delta$ be a d-dimensional cone containing v. The following observations are helpful in computing the motivic zeta function $Z(U_\sigma,I;s)$.

1. For a $\mathcal{L}(T)$-orbit $\mathcal{L}(T)_v \subset \text{Cont}^i(I \cdot \mathcal{O}_Y) \cap \text{Cont}^j(\text{Jac}_h)$, it follows from Proposition 3.1 that
$$i = \min_{x^u \in I \cdot \mathcal{O}_Y(U_\tau)} \langle v, u \rangle \quad \text{and} \quad j = \min_{x^u \in \text{Jac}_h(U_\tau)} \langle v, u \rangle.$$
2. By (5.2) the motivic measure $\mu(\mathcal{L}(T)_v) = (1-\mathbb{L}^{-1})^d \cdot \mathbb{L}^{-\langle v, e_\tau \rangle}$, where e_τ is the sum of primitive generators on the rays of $\tau^\vee$.
3. Denote by $Vert(P_I)$ the set of vertices of the Newton polyhedron P_I of I, then we have
$$\nu_I(v) := \min_{u \in P_I} \langle v, u \rangle = \min_{u \in \text{Vert}(P_I)} \langle v, u \rangle = \min_{x^u \in I \cdot \mathcal{O}_Y(U_\tau) \langle v, u \rangle}.$$
4. Define $\nu'(v) := \langle v, e_\tau \rangle$. Then $\nu'(v) = \min_{x^u \in \Omega_Y^d(U_\tau)} \langle v, u \rangle$. Here, we identify the canonical sheaf Ω_Y^d of the smooth toric variety Y with the torus invariant ideal sheaf whose local sections on U_τ correspond to lattice points in the relative interior of $\tau^\vee$ [9, Section 4.3].
5. Since the sheaf Ω_X^d of Kähler differential d-forms is M-graded [4, Section 3], the image of the toric map $h^*\Omega_X^d(U_\tau) \to \Omega_Y^d(U_\tau)$ is also M-graded. This implies that the Jacobian ideal $\text{Jac}_h(U_\tau)$ is a monomial ideal in the polynomial ring $\mathcal{O}(U_\tau)$. So we can define
$$\nu''(v) := \min_{x^u \in \text{Jac}_h(U_\tau)} \langle v, u \rangle$$
6. Notice that the two valuations ν' and ν'' extend to the function field of U_τ, so they do not depend on the choice of τ. So we put
$$\nu_\Delta(v) := \nu'(v) + \nu''(v) = \min_{x^u \in \Omega_Y^d \cdot \text{Jac}_h(U_\tau)} \langle v, u \rangle.$$

   Again, we abuse the notation Ω_Y^d to denote the ideal sheaf whose local sections on U_τ correspond to lattice points in the relative interior of $\tau^\vee$. Note that the function ν_Δ is linear on each cone of the fan Δ.

With the above observations, we conclude that the motivic zeta function (6.1) of the pair (U_σ, I) can be computed as

$$\begin{aligned} Z(U_\sigma, I; s) &= \sum_{v \in \sigma \cap N} \mu(\mathcal{L}(T)_v) \cdot \mathbb{L}^{-[\nu_I(v)s + \nu''(v)]} \\ &= \sum_{v \in \sigma \cap N} (1-\mathbb{L}^{-1})^d \cdot \mathbb{L}^{-\nu'(v)} \cdot \mathbb{L}^{-[\nu_I(v)s + \nu''(v)]} \\ &= \sum_{v \in \sigma \cap N} (1-\mathbb{L}^{-1})^d \cdot (\mathbb{L}^{-1})^{\nu_I(v)s + \nu_\Delta(v)} \end{aligned} \tag{6.2}$$

Note that the change of variables formula is valid for any log resolution, so the zeta function $Z(U_\sigma, I; s)$ does not depend on the fan Δ. Notice also that the expression (6.2) of $Z(U_\sigma, I; s)$ does generalize the expression in (5.2). Indeed, if σ is a pointed smooth cone, then by taking h to be the identity map

$$\nu_\Delta(v) = \min_{x^u \in \Omega_X^d} \langle v, u \rangle = \min_{u \in \text{int}(\sigma^\vee) \cap M} \langle v, u \rangle = \langle v, e_\sigma \rangle.$$

Since the zeta functions (6.2) and (5.2) have similar expressions, we have the following generalization of the first part of Theorem 5.1 whose proof goes exactly the same as in [10] using their results of zeta functions associated to cones.

Theorem 6.1. The motivic zeta function $Z(U_\sigma, I; s)$ is a rational function of $\mathbb{L}^{-s}$. Each pole of $Z(U_\sigma, I; s)$ is of the form $-\frac{\nu_\Delta(v)}{\nu_I(v)}$, *where* v *is the primitive generator on a ray of a fan* Δ *so that* $X(\Delta) \to U_\sigma$ *is a toric log resolution of* (U_σ, I).

Proof. Rewrite the motivic zeta function (6.2) as

$$Z(U_\sigma, I; s) = (1-\mathbb{L}^{-1})^d \cdot \sum_{\tau\in\Delta_I} Z_{\tau,\nu_I,\nu_\Delta}(s), \text{ where}$$

$$Z_{\tau,\nu_I,\nu_\Delta}(s) := \sum_{v\in \mathrm{int}(\tau)\cap\mathbb{Z}^d} (\mathbb{L}^{-1})^{\nu_I(v)s+\nu_\Delta(v)}.$$

Since ν_I and ν_Δ are linear on each $\tau\in\Delta_I$, the statement of the theorem follows from [10, Theorem 1.3]. □

Next, we investigate whether poles of the motivic zeta function $Z(U_\sigma,I;s)$ are roots of the Bernstein–Sato polynomial $b_I^{U_\sigma}(\mathfrak{s})$ as defined in [11].

Example 6.2. Consider the affine toric variety U_σ whose cone $\sigma = \mathrm{Cone}(e_2, de_1 - e_2)$, $d\geq 2$. The dual cone $\sigma^\vee = \mathrm{Cone}(e_1^*, e_1^* + de_2^*)$. Introduce variables x,y so that the coordinate ring of U_σ is represented as $k[\sigma^\vee \cap M] = k[x, xy, ..., xy^d]$. Let I be the ideal $\langle x^3y^2, x^4y\rangle$ in $k[\sigma^\vee\cap M]$.

Let $h: X(\Delta)\to U_\sigma$ be the minimal log resolution of (U_σ,I), namely $\Delta = \tau_1\cup\tau_2\cup\tau_3$ where $\tau_1 = \mathrm{Cone}(e_1, de_1 - e_2)$, $\tau_2 = \mathrm{Cone}(e_1, e_1 + e_2)$, and $\tau_3 = \mathrm{Cone}(e_1 + e_2, e_2)$. We identify the coordinate ring of the local charts U_{τ_i} of $X(\Delta)$ as

$$k[\tau_1^\vee \cap M] = k[y^{-1}, xy^d],\ k[\tau_2^\vee \cap M] = k[xy^{-1}, y], \text{ and } k[\tau_3^\vee \cap M] = k[x, x^{-1}y].$$

The Jacobian ideals of the map h on the local charts are

$$\mathrm{Jac}_h(U_{\tau_1}) = \langle xy^d\rangle,\ \mathrm{Jac}_h(U_{\tau_2}) = \langle xy\rangle, \text{ and } \mathrm{Jac}_h(U_{\tau_3}) = \langle x^2\rangle.$$

Put $w_1 = (2,2d-1)$ and $w_2 = (2, 1) = w_3$. Then we have

$$\nu_\Delta(v) = \langle v, w_i\rangle \text{ for } i = 1, 2, 3.$$

Put $v_1 = (d,-1)$, $v_2 = (1,0)$, $v_3 = (1,1)$, and $v_4 = (0,1)$. These are primitive generators on the rays of Δ. Note that v_2 comes from the ray of Δ that is not a ray of the normal fan Δ_I. We have $\nu_\Delta(v_1) = 1 = \nu_\Delta(v_4)$, $\nu_\Delta(v_2) = 2$ and $\nu_\Delta(v_3) = 3$. On the other hand, we have $\nu_I(v_1) = 3d - 2$, $\nu_I(v_2) = 3$, $\nu_I(v_3) = 5$, and $\nu_I(v_4) = 1$, so the candidate poles of $Z(U_\sigma,I;s)$ are

$$\frac{-1}{3d-2}, \frac{-2}{3}, \frac{-3}{5}, \text{ and } -1.$$

For $d = 2$, the b-function for the pair (U_σ,I) is

$$b_I^{U_\sigma}(\mathfrak{s}) = (s+1)^2\left(s+\frac{1}{2}\right)^2\left(s+\frac{1}{4}\right)\left(s+\frac{3}{4}\right)\left(s+\frac{2}{5}\right)\left(s+\frac{3}{5}\right)\left(s+\frac{4}{5}\right)\left(s+\frac{6}{5}\right)$$
$$\times\left(s+\frac{7}{10}\right)\left(s+\frac{9}{10}\right)\left(s+\frac{11}{10}\right)\left(s+\frac{13}{10}\right).$$

One sees that $\frac{-1}{4}, \frac{-3}{5}$, and -1 are roots of $b_I^{U_\sigma}(\mathfrak{s})$, but $\frac{-2}{3}$ is not. We point out that b-function algorithms have been developed and implemented, see [1].

In fact, Theorem 6.4 shows that a candidate pole coming from a ray of the normal fan Δ_I is always a root of $b_I^{U_\sigma}(\mathfrak{s})$. Moreover, it follows from [10, Lemma 3.1] that $\frac{-2}{3}$ does appear as a pole of $Z(U_\sigma,I;s)$ (not just a candidate). Indeed, one can verify this by explicitly computing $Z(U_\sigma,I;s)$. Set $T = L^{-s}$ and $B_i := (T)^{\nu_I(v_i)s+\nu_\Delta(v_i)}$. Then by [10, Lemma 3.1], we have

$$
\begin{aligned}
Z(U_\sigma, I; s) &= \sum_{1\le i\le 3} \frac{1}{1-B_i}\cdot\frac{1}{1-B_{i+1}} - \sum_{i=2,3}\frac{1}{1-B_i}\\
&= \frac{(1-B_1B_3)(1-B_4)+B_4(1-B_2)(1-B_1)}{\prod_{1\le i\le 4}(1-B_i)}\\
&= \frac{(1-T^{9s+4})(1-T^{s+1})+T^{s+1}(1-T^{3s+2})(1-T^{4s+1})}{(1-T^{4s+1})(1-T^{3s+2})(1-T^{5s+3})(1-T^{s+1})}.
\end{aligned}
$$

Example 6.3. Even for principal ideals, the situation is not as expected. Consider $\sigma = \mathrm{Cone}(3e_1 - 2e_2, e_2)$ and $I = \langle x^3y^2\rangle$ a principal ideal on U_σ whose coordinate ring is represented as $k[x, xy, x^2y^3]$. Let $h: X(\Delta)\to U_\sigma$ be the minimal log resolution of (U_σ, I), namely $\Delta = \tau_1\cup\tau_2\cup\tau_3$ where $\tau_1 = \mathrm{Cone}(3e_1 - 2e_2, 2e_1 - e_2)$, $\tau_2 = \mathrm{Cone}(2e_1 - e_2, e_1)$, and $\tau_3 = \mathrm{Cone}(e_1, e_2)$. We identify the coordinate ring of the local charts U_{τ_i} of $X(\Delta)$ as

$$k[\tau_1^\vee\cap M] = k[x^2y^3, x^{-1}y^{-2}],\ k[\tau_2^\vee\cap M] = k[xy^2, y^{-1}],\ \text{and}\ k[\tau_3^\vee\cap M] = k[x, y].$$

The Jacobian ideals of the map h on the local charts are

$$\mathrm{Jac}_h(U_{\tau_1}) = \langle x^2y^3\rangle,\ \mathrm{Jac}_h(U_{\tau_2}) = \langle x, x^2y^3\rangle,\ \text{and}\ \mathrm{Jac}_h(U_{\tau_3}) = \langle x\rangle.$$

One verifies that the minimal log resolution h produces three candidate poles of $Z(U_\sigma, I; s)$, $\frac{-1}{5}, \frac{-1}{2}, \frac{-2}{3}$. Only $\frac{-2}{3}$ is not a root of $b_I^{U_\sigma}(\mathfrak{s}) = (s+1)^2(s+\frac{1}{2})(s+\frac{1}{5})(s+\frac{2}{5})(s+\frac{3}{5})(s+\frac{4}{5})$.

Theorem 6.4. *Let $h: X(\Delta)\to U_\sigma$ be a toric log resolution of (U_σ, I) such that $I\cdot\mathscr{O}_{X(\Delta)}\subseteq Jac_h\cdot\Omega_{X(\Delta)}$. Here, we identify $\Omega_{X(\Delta)}$ with the torus invariant ideal of $\mathscr{O}_{X(\Delta)}$ whose local sections on each chart U_τ, $\tau\in\Delta$, correspond to the lattice points in the interior of $\tau^\vee$. If v is a primitive generator on a ray of Δ_I such that $\nu_I(v)\neq 0$, then $-\frac{\nu_\Delta(v)}{\nu_I(v)}$ is a root of the Bernstein–Sato polynomial $b_I^{U_\sigma}(\mathfrak{s})$.*

Proof. We use freely the notation in Section 4. Let v be a primitive generator on a ray of Δ_I such that $\nu_I(v)\neq 0$. Let Q be the facet of the Newton polyhedron P_I that corresponds to v. Note that Q is not contained in any faces of σ. Note also that the assumption $I\cdot\mathscr{O}_{X(\Delta)}\subseteq \mathrm{Jac}_h\cdot\Omega_{X(\Delta)}$ implies that $\nu_I(v)\ge\nu_\Delta(v)$. Let τ be a d-dimensional cone of Δ that contains v and let $w\in\tau^\vee\cap M$ be a lattice point that corresponds to a monomial in the ideal $\mathrm{Jac}_h\cdot\Omega_{X(\Delta)}(U_\tau)$ so that $\nu_\Delta = \langle v, w\rangle$. We have $\nu_I(v)\ge\langle v, w\rangle\ge 1$.

Let $\tilde{v}\in\Sigma\cap\mathbb{Q}^n$ be such that $F^*(\tilde{v}) = v$ where $F^*: (\mathbb{R}^n)^*\to(\mathbb{R}^d)^*$ is the dual of the map F. Then

$$\langle v, u\rangle = \langle F^*(\tilde{v}), u\rangle = \langle\tilde{v}, F(u)\rangle,\quad \text{for } u\in\mathbb{Z}^d.$$

This implies that $\nu_I(v) = \min_{x^u\in I}\langle\tilde{v}, F(u)\rangle = \nu_J(\tilde{v})$ and that

$$-\frac{\nu_\Delta(v)}{\nu_I(v)} = -\frac{\langle v, w\rangle}{\nu_J(\tilde{v})} = -\frac{\langle\tilde{v}, F(w)\rangle}{\nu_J(\tilde{v})}.$$

Set $L(a) := \frac{\langle\tilde{v}, a\rangle}{\nu_J(\tilde{v})}$. Then $L(a)$ takes the value 1 on a face $\tilde{Q}$ containing $F(Q)$. By Theorems 4.1 and 4.2, it suffices to show that $F(w)\in(M_{\tilde{Q}}\setminus M'_{\tilde{Q}})\cap V_{\tilde{Q}}$. Since Q is a facet of P_I, it is certainly true that $w\in V_Q$ and hence $F(w)\in F(V_Q)\subseteq V_{\tilde{Q}}$. Moreover, pick any $b\in\Gamma_J\cap\tilde{Q}$. Since w lies in the relative interior of σ, we have $F(w)-\mathbb{1}\in int\Sigma$ and $a := F(w)-\mathbb{1}+b\in\Gamma_J$. Hence $F(w) = \mathbb{1}+a-b\in M_{\tilde{Q}}$.

To show that $F(w)\notin M'_{\tilde{Q}}$, it suffices to observe that the linear functional L takes values that are strictly greater than 1 on $M'_{\tilde{Q}}$, but $L(F(w)) = \frac{\langle v, w\rangle}{\nu_I(v)}\le 1$. □

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