Solutions for Calculus Midterm 2

(1) (i) Since the management sets a production goal of 20000 units, the constraint is

\[ f(x, y) = 100x^{0.25}y^{0.75} = 20000. \]

That is

\[ x^{0.25}y^{0.75} - 200 = 0. \]

And we want to minimize the cost \(48x + 36y\).

Then we use the method of Lagrange Multiplies and begin by writing the function

\[ F(x, y, \lambda) = 48x + 36y - \lambda(x^{0.25}y^{0.75} - 200). \]

Set

\[ F_x = 48 - 0.25\lambda(y/x)^{0.75} = 0, \]
\[ F_y = 36 - 0.75\lambda(x/y)^{0.25} = 0, \]
\[ F_\lambda = 200 - x^{0.25}y^{0.75} = 0. \]

This implies \( \frac{y}{x} = 4 \).

Thus \( x = 50\sqrt{2} \) and \( y = 200\sqrt{2} \).

(ii)

\[
\begin{align*}
\text{Marginal productivity of labor} & = \frac{f_x(x, y)}{f_y(x, y)} \\
& = \frac{25x^{-0.75}y^{0.75}}{75x^{0.25}y^{-0.25}} \\
& = \frac{1}{3} \left( \frac{y}{x} \right) \\
& = \frac{4}{3} \text{ by (i)} \\
& = \frac{48}{36} \text{ unit price of labor} \\
& = \frac{\text{unit price of labor}}{\text{unit price of capital}}.
\end{align*}
\]
(2) (i) The sum of squared errors of \( y = ax + b \) with respect to the \( n \) points is

\[
S = [(ax_1 + b) - y_1]^2 + [(ax_2 + b) - y_2]^2 + \ldots + [(ax_n + b) - y_n]^2
\]

\[
= \sum_{i=1}^{n} (ax_i + b - y_i)^2.
\]

(ii)

\[
A = \frac{n \sum xy - (\sum x)(\sum y)}{n \sum x^2 - (\sum x)^2}
\]

\[
= \frac{4 \times 91 - 13 \times 21}{4 \times 51 - 169}
\]

\[
= 2.6
\]

\[
B = \frac{1}{n} (\sum y - A \sum x)
\]

\[
= \frac{1}{4} (21 - 2.6 \times 13)
\]

\[
= -3.2
\]

Thus the least squares regression line is \( y = 2.6x - 3.2 \).
(3) 

(i) We show it in Figure 1.

![Figure 1: The region $R$ on $x - y$ plane.](image)

(ii) 

\[
\int \int_{R} \frac{y}{x^2 + y^2} \, dA = \int_{0}^{2} \int_{x}^{2} \frac{y}{x^2 + y^2} \, dy \, dx \\
\int \int_{R} \frac{y}{x^2 + y^2} \, dA = \int_{0}^{2} \int_{\frac{y}{2}}^{y} \frac{y}{x^2 + y^2} \, dx \, dy + \int_{2}^{4} \int_{\frac{y}{2}}^{2} \frac{y}{x^2 + y^2} \, dx \, dy
\]
\[ \int_0^2 \int_x^{2x} \frac{y}{x^2 + y^2} dy \, dx = \int_0^2 \left( \frac{1}{2} \ln(x^2 + y^2) \right)_{y=x}^{y=2x} dx \\
= \int_0^2 \frac{1}{2} (\ln 2x^2 - \ln 2x^2) dx \\
= \int_0^2 \frac{1}{2} \ln \frac{5x^2}{2x^2} dx \\
= \frac{1}{2} \int_0^2 \ln \frac{5}{2} dx \\
= \frac{1}{2} \ln \frac{5}{2} \left|_0^2 \right. \\
= \ln \frac{5}{2} \]

(4) (i) \( f(t) = \sin\left(\frac{2\pi t}{24}\right) \) is a periodic function with period 24.

\[ h(t) = (\sin t)^2 = \frac{1 - \cos 2t}{2} \] is also a periodic function with period \( \pi \).

(ii) We sketch those graphs in Figure 2 and Figure 3.
Figure 2: $f(t) = \sin\left(\frac{2\pi t}{24}\right)$.

Figure 3: $h(t) = \sin^2 t$. 
(5) $F(t)$ attains minimum when $\sin \frac{2\pi(t - 30)}{365} = -1.$

$$\Rightarrow \frac{2\pi(t - 30)}{365} = \frac{3\pi}{2}.$$  

$$\Rightarrow t = 303.75 \approx 304.$$  

Thus the minimum sales occur on the 304st day of the year, that is on October 31.

(6)

$$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0.$$  

$$\lim_{x \to \infty} \frac{x^{0.001}}{x} = \lim_{x \to \infty} \frac{0.001x^{-0.999}}{1} = 0.$$  

$$\lim_{x \to \infty} \frac{\ln x}{x^{0.001}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{0.001x^{-0.999}} = 0.$$  

$$\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x \to \infty} \frac{\frac{1}{x} \frac{1}{x}}{\ln x} = \lim_{x \to \infty} \frac{1}{x} = 0.$$  

Thus we order them as $g(x) = e^x, f(x) = x, I(x) = x^{0.001}, h(x) = \ln x, J(x) = \ln(\ln x)$.

(i) $g(x) = e^x$ grows to infinity with the most rapid rate.

(ii) $J(x) = \ln(\ln x)$ grows to infinity with the slowest rate.

(iii) The middle is $I(x) = x^{0.001}$.

(7) (i)

$$\int_0^1 \tan(1 - x)dx = \int_0^1 \frac{\sin(1 - x)}{\cos(1 - x)}dx$$  

$$= \ln |\cos(1 - x)|_0^1$$  

$$= \ln 1 - \ln(\cos 1)$$  

$$= - \ln(\cos 1)$$

(ii) Let $u = x, dv = \cos x dx$. Then $du = dx$ and $v = \sin x$. Apply integration by parts and obtain

$$\int x \cos x dx = x \sin x - \int \sin x dx$$  

$$= x \sin x + \cos x + C.$$