Solutions for Calculus Quiz #2

1. (a) \[
\lim_{x \to 0} \frac{e^{2x} - 1}{e^x - 1} = \lim_{x \to 0} \frac{(e^x + 1)(e^x - 1)}{e^x - 1} = \lim_{x \to 0} e^x + 1 = e^0 + 1 = 2
\]
(b) The degree of the numerator is greater than that of the denominator, so the limit does not exist. And \[
\frac{2 + x^2}{1 - x}
\]
behaves as \[
\frac{x^2}{-x} = -x
\]
when \(x \to -\infty\).

Thus \[
\lim_{x \to -\infty} \frac{2 + x^2}{1 - x} = \infty.
\]
(c) We know that \(-1 \leq \sin x \leq 1\).

Then \[
\frac{-1}{|x|} \leq \frac{\sin x}{x} \leq \frac{1}{|x|}.
\]
Also \[
\lim_{x \to -\infty} \frac{1}{|x|} = \lim_{x \to -\infty} \frac{1}{|x|} = 0.
\]
By Sandwich Theorem, we obtain that \[
\lim_{x \to -\infty} \frac{\sin x}{x} = 0.
\]
(d) \[
\lim_{x \to 0} \frac{\sin x \cos x}{x(1 - x)} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\cos x}{1 - x} = (\lim_{x \to 0} \frac{\sin x}{x})(\lim_{x \to 0} \frac{\cos x}{1 - x}) = 1 \cdot 1 = 1.
\]

2. (a) Let \(g(x) = e^x\), \(h(x) = -|x|\). Then \(g(x)\) is differentiable and continuous. \(h(x)\) is continuous but it is not differentiable at \(x = 0\). Thus \(f(x) = (g \circ h)(x)\) is continuous for \(x \in \mathbb{R}\), and is differentiable for \(x \in \mathbb{R}\setminus\{0\}\).

(b) \[
\frac{x}{x + 1} > 0 \Rightarrow x < -1, x > 0.
\]
Thus the domain of \(f(x)\) is \(D = \{x \in \mathbb{R} | x < -1, x > 0\}\), and it is continuous and differentiable in \(D\). \((f(x)\) is not continuous and not differentiable at \(x \in [-1, 0]\).\)

3. Let \(k \in \mathbb{Z}\).

Consider \(\varepsilon = \frac{1}{2}\). If \(f(x)\) is continuous at \(x = k\), then \(\exists \delta > 0\) such that \(\forall x \in (k - \delta, k + \delta) \Rightarrow |f(x) - k| < \varepsilon = \frac{1}{2}\). But for \(x \in (k - \delta, k)\), \(f(x) = k - 1\), this implies that \(|f(x) - k| = 1 > \frac{1}{2}\). Contradiction! Thus \(f(x)\) is not continuous at \(x = k\).

4. Let \(f(x) = \sin x - x\).

A solution of \(\sin x = x\) is a value of \(x\) for which \(f(x) = 0\).

\(f(1) = \sin 1 - 1 < 0\), \(f(-1) = \sin(-1) - (-1) = 1 - \sin 1 > 0\) since \(|\sin x| < 1\) except for \(x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}\), ...

And \(f(x)\) is continuous, we apply the Intermediate Value Theorem, there exists \(c \in (-1, 1)\) such that \(f(c) = 0\). That is, \(\sin c = c\).
5. Let \( f(x) = |x| \). Then \( f(x) \) is continuous. See page 174, Example 4 in the textbook, \( f(x) \) is not differentiable at \( x = 0 \).

6. Let \( y = f(x) = -e^2x^2 - ex \).
   Then \( f'(x) = -2e^2x - e \)
   \( \Rightarrow f'(0) = -e \).
   The slope of the normal line is \( -\frac{1}{f'(0)} = \frac{1}{e} \).
   Thus the equation of the normal line is
   \[
   y - f(0) = \frac{1}{e}(x - 0)
   \]
   \( \Rightarrow y = \frac{1}{e}x. \)

7. \( f'(R) = \frac{a(k + R) - aR \cdot 1}{(k + R)^2} = \frac{ak}{(k + R)^2}. \)
   \( \therefore f'(N) = \frac{ak}{(k + N)^2}. \)
   \( f(R) \geq 0 \) and \( f'(R) \geq 0 \) for \( R \geq 0 \).
   \[
   \lim_{R \to \infty} f(R) = \lim_{R \to \infty} \frac{aR}{k + R} = \lim_{R \to \infty} \frac{a}{R} + 1 = a.
   \]
   \( f(0) = 0, f'(0) = \frac{a}{k} > a. \)
   \[
   \lim_{R \to \infty} f'(R) = \lim_{R \to \infty} \frac{ak}{(k + R)^2} = 0.
   \]
   We graph \( f(R) \) and \( f'(R) \) in the next page.

8. We have
   \[
   (\frac{f}{2g})' = \frac{1}{2} (\frac{f}{g})' = \frac{1}{2} \frac{f'g - fg'}{g^2}
   \]
   Thus
   \[
   (\frac{f}{2g})'(2) = \frac{1}{2} \frac{1 \cdot 3 - (-4)(-2)}{3^2} = -\frac{5}{18}.
   \]