No credit will be given for an answer without reasoning.

1. 
   (i) [5\%] Give an example of a nondegenerate symmetric bilinear form of Witt index 1 on a two-
dimensional real vector space.
   (ii) [5\%] Give an example of nonzero quadratic form on a two-dimensional real vector space.

2. Let $H$ be the quaternion algebra over $\mathbb{R}$ and let $M_2(\mathbb{R})$ be the matrix algebra of two by two real matrices.
   (i) [5\%] Is $H$ isomorphic to $M_2(\mathbb{R})$ as vector spaces over $\mathbb{R}$? Why or why not?
   (ii) [5\%] Is $H$ isomorphic to $M_2(\mathbb{R})$ as $\mathbb{R}$-algebras? Why or why not?

3. Let $V = F^2$ be a two-dimensional vector space over a field $F$. Define two unit vectors $i = (1, 0)$ and $j = (0, 1)$. It is obvious that $\{i, j\}$ is a basis for $V$. Define $f: V \times V \to F$ by
   $f((a_1, b_1), (a_2, b_2)) = 2b_1b_2$
   for $a_1, a_2, b_1, b_2 \in F$.
   (i) [5\%] What is the radical of $V$?
   (ii) [5\%] Suppose that the characteristic of $F$ is not 3. Find the matrix presenting the form $f$ with respect to the basis $\{9i, 3j\}$.

4. [10\%] Find the Jordan canonical form of the matrix
   \[
   \begin{bmatrix}
   0 & 1 & 1 \\
   0 & 0 & 1 \\
   0 & 0 & 0
   \end{bmatrix}
   \]

5. Let $V = \mathbb{R}^2$ be a two-dimensional real vector space. Fix a basis $\{i, j\}$ for $V$ where $i = (1, 0)$ and $j = (0, 1)$. Define
   $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2$
   for $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
   (i) [5\%] Show that the linear transformation
   \[
   \begin{bmatrix}
   \cos \theta & \sin \theta \\
   -\sin \theta & \cos \theta
   \end{bmatrix}
   \]
   where $\theta$ is a real number is an isometry of $V$.
   (ii) [5\%] Suppose that $\tau: V \to V$ is an isometry. Show that $a\tau$ for $a \in \mathbb{R}$ is an isometry if and only if $a = \pm 1$. 

6. Let $F$ be a field and $V$ be a vector space over $F$. Suppose that $f$ is a bilinear form on $V$.
   (i) [5%] Show that if $f$ is alternating, then $f$ is skew-symmetric.
   (ii) [5%] Show that if $f$ is skew-symmetric and the characteristic of $F$ is not 2, then $f$ is alternating.

7. Let $V = \mathbb{C}^2$ be a two-dimensional vector space over $\mathbb{C}$. Define a symmetric bilinear form $f$ on $V$ by
   \[ f((x_1, x_2), (y_1, y_2)) = x_1y_1 + x_2y_2 \]
   for $x_1, x_2, y_1, y_2 \in \mathbb{C}$.
   (i) [5%] Show that $f$ is isotropic.
   (ii) [5%] Show that $V$ is a hyperbolic plane.

8. Let $V = \mathbb{R}^2$ be a two-dimensional vector space over a field $\mathbb{R}$ with a nondegenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by
   \[ \langle (x_1, y_1), (x_2, y_2) \rangle = x_1y_2 - y_1x_2 \]
   for $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
   (i) [5%] Suppose that $f$ is a linear functional on $V$. The Riesz representation theorem tells us that there is a unique element $x \in V$ such that $f = \phi_x$ where $\phi_x \in V^*$ is defined by $\phi_x(v) = \langle v, x \rangle$.
      Find $x$ for the linear functional $f$ defined by $f((x, y)) = 2x + 3y$ for $x, y \in \mathbb{R}$.
   (ii) [5%] Let $S$ be the subspace spanned by the vector $(1, 0)$. Is $V = S \perp S^\perp$? Why or why not?

9. [10%] Let $P_2 \subset \mathbb{R}[x]$ be the space of polynomials of degree less than or equal to 2. Define
   \[ \langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx \]
   for $f, g \in P_2$. We know that $\{1, x, x^2\}$ is a basis for $P_2$. Apply Gram-Schmidt orthogonalization process to $\{1, x, x^2\}$ to find an orthogonal basis for $P_2$.

10. Let $\ell^2$ be the set of all real infinite sequences $(a_n)$ such that $\sum_{n=1}^{\infty} |a_n|$ is finite. Define $(a_n) + (b_n) = (a_n + b_n)$ and $r(a_n) = (ra_n)$ for $(a_n), (b_n) \in \ell^2$ and $r \in \mathbb{R}$.
    (i) [5%] Show that $\ell^2$ is a vector space over $\mathbb{R}$.
    (ii) [5%] Show that $\ell^2$ is an inner product space under the inner product $\langle \cdot, \cdot \rangle$ defined by
        \[ \langle (a_n), (b_n) \rangle = \sum_{n=1}^{\infty} a_nb_n. \]