Each of the following problems worth 10 points.

1. (i) Give an example of a group of order 4 which is not cyclic.
    (ii) Give an example of an infinite non-abelian group.
    (iii) Give an example of a non-abelian solvable group.
    (iv) Give an example of a non-commutative division ring.
    (v) Give an example of an ideal \( I \) of a commutative ring \( R \) such that \( I \) is prime but not maximal.

2. (i) What is the characteristic of the ring \( \mathbb{Z}_6 \times \mathbb{Z}_7 \)? Why?
    (ii) What is the commutator subgroup of a simple non-abelian group? Why?
    (iii) What is the order of the element \((12)(345)(12)\) in \( S_8 \)? Why?

3. Suppose that \( H \) is a normal subgroup of a group \( G \) and \( K \) is a normal subgroup of \( H \). Let \( a \) be an element in \( G \).
   (i) Show that \( aKa^{-1} \subseteq H \).
   (ii) Show that \( aKa^{-1} \) is a normal subgroup of \( H \).

4. (i) Find all prime number \( p \) such that \( x + 2 \) is a factor of \( x^4 + x^3 + x^2 - x + 1 \) in \( \mathbb{Z}_p[x] \).
    (ii) Show that for \( p \) a prime, the polynomial \( x^p + a \) in \( \mathbb{Z}_p[x] \) is not irreducible for any \( a \in \mathbb{Z}_p \).

5. Show that \( \phi: \mathbb{C} \rightarrow M_2(\mathbb{R}) \) given by
   \[
   \phi(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}
   \]
   for \( a, b \in \mathbb{R} \) gives an isomorphism of \( \mathbb{C} \) with the subring \( \phi[\mathbb{C}] \) of \( M_2(\mathbb{R}) \) where \( M_2(\mathbb{R}) \) is the ring of two by two matrices over \( \mathbb{R} \).

6. (i) Is \( \mathbb{Q}[x]/\langle x^2 - 5x + 6 \rangle \) a field? Why?
    (ii) Is \( \mathbb{Q}[x]/\langle x^2 - 6x + 6 \rangle \) a field? Why?

7. Let \( A \) and \( B \) be ideals of a ring \( R \). The product \( AB \) of \( A \) and \( B \) is defined by
   \[
   AB = \left\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in A, \ b_i \in B, \ n \in \mathbb{Z}^+ \right\}.
   \]
   (i) Show that \( AB \) is an ideal of \( R \).
   (ii) Show that \( AB \subseteq (A \cap B) \).

8. Let \( R \) be a commutative ring and \( N \) be an ideal of \( R \). Define
   \[
   \sqrt{N} = \{ a \mid a^n \in N \text{ for some } n \in \mathbb{Z}^+ \}.
   \]
   (i) Show that \( N \subseteq \sqrt{N} \) and \( \sqrt{N} \) is an ideal of \( R \).
   (ii) Give an example of \( N \) such that \( \sqrt{N} = N \).
(ii) Give an example of $N$ such that $\sqrt{N} \neq N$.

9.  
(i) Let $K$ be a subgroup of index 2 of a group $G$. Suppose that $a \in G - K$ and $b \in G - K$ i.e., $a, b$ are in $G$ but not in $K$. Show that $ab \in K$.
(ii) Let $G$ be a finite abelian group. Suppose that $G$ has two distinct elements of order 2. Show that 4 divides $|G|$.

10. Let $\phi: \mathbb{R} \to \mathbb{R}$ be a nontrivial ring homomorphism.
   (i) Show that $\phi(a) = a$ if $a \in \mathbb{Z}$.
   (ii) Show that $\phi(a) = a$ if $a \in \mathbb{Q}$.
   (iii) Show that $\phi(\mathbb{R}^+) \subseteq \mathbb{R}^+$ where $\mathbb{R}^+ = \{ a \in \mathbb{R} \mid a > 0 \}$. (Hint: a square is positive.)
   (iv) Show that $\phi(a) > \phi(b)$ if $a, b \in \mathbb{R}$ and $a > b$.
   (v) Show that $\phi(a) = a$ for all $a \in \mathbb{R}$. 