Final Exam of Calculus

Date: 2000, June 19, 13:10–14:55

An answer without reasoning will not be accepted.

1. [8%] Find curl $\mathbf{F}$ and div $\mathbf{F}$ if $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 2x\sin y\mathbf{j} + 2z\cos y\mathbf{k}$.

2. [8%] Show that there is no vector field $\mathbf{G}$ such that $\text{curl}\, \mathbf{G} = 2x\mathbf{i} + 3yz\mathbf{j} - xz^2\mathbf{k}$.

3. [10%] Compute the outward flux of $\mathbf{F}(x, y, z) = xi + yj + zk$ through the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

4. [10%] Compute the surface integral

$$\iint_{S} (x^2 + y^2)\, d\sigma,$$

where $S$ is the hemisphere $z = \sqrt{1 - (x^2 + y^2)}$.

5. [14%] Let $\Omega$ be a Jordan region (on the $xy$-plane) with a piece-wise smooth boundary $C$, and let $f$ and $g$ be continuously differentiable functions on an open set containing $\Omega$.

   (i) Use the vector form of the Green’s theorem to prove Green first identity:

   $$\iint_{\Omega} f\nabla^2 g\, d\sigma = \oint_{C} f\nabla g \cdot \mathbf{n}\, ds - \iint_{\Omega} \nabla f \cdot \nabla g\, d\sigma$$

   where $\mathbf{n}$ is the outer unit normal vector.

   (ii) Use above Green’s first identity to prove Green’s second identity:

   $$\iint_{\Omega} (f\nabla^2 g - g\nabla^2 f)\, d\sigma = \oint_{C} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, ds.$$

6. [12%] Show that the vector field $\mathbf{v}(x, y, z) = (2xz + \sin y)i + x\cos yj + x^2k$ is a gradient. Then evaluate the line integral of $\mathbf{v}$ over the curve $\mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j} + u\mathbf{k}$ for $u \in [0, \pi]$.

7. [10%] Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $x^2 + 4y^2 + z^2 = 36$.

8. [6%] Compute the sum of the series

$$1 - \ln \pi + \frac{(\ln \pi)^2}{2!} - \frac{(\ln \pi)^3}{3!} + \cdots$$

9. [8%] Compute the interval of convergence of the Taylor series in $x$ of $\ln(1 - x)$.

10. [8%] Find the tangential component of the acceleration vector of $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$.

11. [8%] A function $f(x, y)$ is called homogeneous of degree $n$ if $f$ has continuous second-order partial derivatives and $f(tx, ty) = t^n f(x, y)$ for all $t$. Use the chain rule to show that if $f$ is homogeneous of degree $n$, then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y).$$