**FINAL FOR CALCULUS**

**Time:** 8:10–10:00 AM, Friday, January 12, 2000  
**Instructor:** Shu-Yen Pan

No calculator is allowed. No credit will be given for an answer without reasoning.

1. (1) [4%] Find \( y' \) for \( y = \sqrt{x + \sqrt{x}} \).
   (2) [4%] Is \( \frac{d}{dx}|x^2 + x| = |2x + 1|? \) Why or why not?

2. (1) [4%] Evaluate \( \int e^{x+x^2} \, dx \).
   (2) [4%] Evaluate \( \int_0^1 \ln x \, dx \).

3. (1) [4%] Differentiating the equation \( \tan y = x \) implicitly to find \( \frac{dy}{dx} \).
   (2) [4%] One model for the spread of a rumor is that the rate of the spread is proportional to the product of the fraction \( y \) of the population who have heard the rumor and the faction who have not heard the rumor. Write a differential equation that is satisfied by \( y \).

4. A spinner from a board game randomly indicates a real number between 0 and 10. The spinner is fair in the sense that it indicates a number in a given interval with the same probability as it indicates a number in any other interval of the same length.
   (1) [4%] Explain why the function \( f(x) = \begin{cases} 0.1 & \text{if } 0 \leq x \leq 10; \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases} \) is a probability density function for the spinner’s values.
   (2) [4%] What does your intuition tell you about the value of the mean? Check your answer by evaluating an integral.

5. [6%] Find the arc length function for the curve \( y = 2x^{3/2} \) with starting point \( P_0(1, 2) \).

6. [6%] If \( \lim_{x \to 1} (f(x) + g(x)) = 2 \) and \( \lim_{x \to 1} (f(x) - g(x)) = 6 \), find \( \lim_{x \to 1} f(x)g(x) \).

7. [8%] If \( f \) is a positive function and \( f''(x) > 0 \) for \( a \leq x \leq b \), show that \( M_n \leq \int_a^b f(x) \, dx \leq T_n \) where \( M_n \) is the approximation by midpoint rule and \( T_n \) is the approximation by trapezoidal rule.

8. [8%] Find \( A \) and \( B \) given that the function \( y = Ax^{-1/2} + Bx^{1/2} \) has a minimum value 6 at \( x = 9 \).

9. [8%] Let \( f \) be a one-to-one function and \( f''(x) \) exists for all \( x \). Let \( g = f^{-1} \). Show that \( g''(x) = -\frac{f''(g(x))}{(f'(g(x)))^2} \).

10. [8%] Show that the area of a sphere of radius \( r \) is \( 4\pi r^2 \).

11. [8%] Find all functions \( f \) that satisfy the equation \( \left( \int f(x) \, dx \right) \left( \int \frac{1}{f(x)} \, dx \right) = -4 \).

12. [8%] A student forgot the product rule for differentiation and made the mistake of thinking that \( (fg)' = f'g' \). However, she was lucky and got the correct answer. The function \( f \) that she used was \( f(x) = e^{x^2} \) and the domain of her problem was the interval \( (\frac{1}{2}, \infty) \). What was the function \( g \)?

13. [8%] Evaluate \( \lim_{x \to 2} \left( \frac{x}{x-2} \int_2^x e^t \, dt \right) \).