Exam Set: C

No credit will be given for an answer without reasoning.

1.  
   (i) [5%] Find the constant a such that the function \( f(x) = ax^2(1 - x) \) is a probability density function on the interval \([0, 1]\).

   (ii) [5%] Keep the assumption in (i). Find the probability \( P(0 \leq x \leq 1/2) \).

2.  
   (i) [5%] Find \( f_y(8, -6) \) for \( f(x, y) = \sqrt{x^2 + y^2} \).

   (ii) [5%] Find the limit \( \lim_{x \to \infty} \frac{x}{\ln x} \).

3.  
   (i) [5%] A ball is dropped from a height of 16 meters. Each time it drops \( h \) meters, it rebounds 0.6\( h \) meters. Find the total vertical distance travelled by the ball.

   (ii) [5%] Determine the convergence or divergence of the series \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} \).

4. Consider the differential equation: \( y' = y(1 - y) \).

   (i) [5%] Find the general solution \( y(t) \).

   (ii) [5%] Find the solution for the equation with initial condition \( y(0) = 1/2 \).

5.  
   (i) [5%] Evaluate the integral \( \int \frac{\sin t \cos t}{\sqrt{1 + \cos^2 t}} \, dt \).

   (ii) [5%] Find the improper integral \( \int_{-\infty}^{\infty} \frac{e^x}{(1 + e^x)^2} \, dx \).

6.  
   (i) [5%] Evaluate the double integral by changing the order of integration:

   \[ \int_0^1 \int_{\sqrt{x}}^1 \sin \left( \frac{y^3 + 1}{2} \right) \, dy \, dx. \]

   (ii) [5%] Let \( f \) be a differentiable function such that \( f(0) = 4, f(3) = 5 \), and \( f'(3) = 5 \). What is \( \int_0^3 xf''(x) \, dx? \)
7. [10%] Assume that the temperature $T$ at a point $(x, y, z)$ on the sphere $x^2 + y^2 + z^2 = 1$ is given by

$$T(x, y, z) = 8xy^2z.$$ 

Use Lagrange multiplier method to find the point(s) on the sphere at which the temperature is greatest and the point(s) at which it is least.

8. Estimate $\sqrt[3]{67}$ by the following two methods:

(i) [5%] Apply Newton’s method for two iterations to the function $f(x) = x^3 - 67$ with initial guess $x_1 = 4$.

(ii) [5%] Use the degree-two Taylor polynomial centered at $x = 64$ to approximate the function $g(x) = x^{1/3}$. Then evaluate the Taylor polynomial at $x = 67$.

9.

(i) [5%] Consider a random variable $X$ with probability

$$P(X = k) = e^{-4} \frac{4^k}{k!}$$

for $k = 0, 1, 2, 3, \ldots$. Find the expected value $E[X]$.

(ii) [5%] Find the standard deviation $\sigma$ of the probability density function $f(x) = 4e^{-4x}$, $0 \leq x < \infty$.

10. [10%] Find the Taylor series centered at $x = 0$ of the function

$$f(x) = \ln \left( \frac{1 + x}{1 - x} \right).$$

Then find the radius of convergence of the Taylor series. [Hint: $f(x) = \ln(1 + x) - \ln(1 - x)$]