Exam Set: D

No credit will be given for an answer without reasoning.

1. (i) [5%] Evaluate the integral
\[ \int \frac{\sin t \cos t}{\sqrt{1 + \cos^2 t}} dt. \]
(ii) [5%] Find the improper integral
\[ \int_{-\infty}^{\infty} \frac{e^x}{(1 + e^x)^3} dx. \]

2. (i) [5%] Find \( f_y(6, 8) \) for \( f(x, y) = \sqrt{x^2 + y^2} \).
(ii) [5%] Find the limit
\[ \lim_{x \to \infty} \frac{\ln x}{x}. \]

3. (i) [5%] Evaluate the double integral by changing the order of integration:
\[ \int_0^1 \int_{\sqrt{x}}^1 \sin \left( \frac{y^3 + 1}{2} \right) dy dx. \]
(ii) [5%] Let \( f \) be a differentiable function such that \( f(0) = 5 \), \( f(3) = 5 \), and \( f'(3) = 4 \). What is \( \int_0^3 x f''(x) \, dx \)?

4. Consider the differential equation: \( y' = y(1 - y) \).
   (i) [5%] Find the general solution \( y(t) \).
   (ii) [5%] Find the solution for the equation with initial condition \( y(0) = 1/3 \).

5. (i) [5%] Consider a random variable \( X \) with probability
\[ P(X = k) = e^{-2} \frac{2^k}{k!} \]
for \( k = 0, 1, 2, 3, \ldots \). Find the expected value \( E[X] \).
(ii) [5%] Find the standard deviation \( \sigma \) of the probability density function \( f(x) = 5e^{-5x}, 0 \leq x < \infty \).

6. (i) [5%] Find the constant \( a \) such that the function \( f(x) = ax^2(2 - x) \) is a probability density function on the interval [0, 2].
(ii) [5%] Keep the assumption in (i). Find the probability \( P(0 \leq x \leq 1/2) \).
7. 
(i) [5%] A ball is dropped from a height of 18 meters. Each time it drops $h$ meters, it rebounds $0.7h$ meters. Find the total vertical distance travelled by the ball.
(ii) [5%] Determine the convergence or divergence of the series
\[ \sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k^2}}. \]

8. Estimate $\sqrt[3]{66}$ by the following two methods:
(i) [5%] Apply Newton’s method for two iterations to the function $f(x) = x^3 - 66$ with initial guess $x_1 = 4$.
(ii) [5%] Use the degree-two Taylor polynomial centered at $x = 64$ to approximate the function $g(x) = x^{1/3}$. Then evaluate the Taylor polynomial at $x = 66$.

9. [10%] Assume that the temperature $T$ at a point $(x, y, z)$ on the sphere $x^2 + y^2 + z^2 = 1$ is given by
\[ T(x, y, z) = 8x^2yz. \]
Use Lagrange multiplier method to find the point(s) on the sphere at which the temperature is greatest and the point(s) at which it is least.

10. [10%] Find the Taylor series centered at $x = 0$ of the function
\[ f(x) = \ln \left( \frac{1 + x}{1 - x} \right). \]
Then find the radius of convergence of the Taylor series. [Hint: $f(x) = \ln(1 + x) - \ln(1 - x)$]