CALCULUS Final SOLUTION

Exam Set: B

1. (i) To find the first partial derivative with respect to \( y \), hold \( x \) constant to obtain

\[
f_y(x, y) = \frac{\partial}{\partial y} \sqrt{x^2 + y^2} = \frac{2y}{2 \sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}
\]

The value of \( f_y(x, y) \) at the point \((8, -6)\) is

\[
f_y(6, 8) = \frac{8}{\sqrt{(6)^2 + (8)^2}} = \frac{8}{10} = \frac{4}{5} = 0.8
\]

(ii) Since \( \lim_{x \to \infty} \frac{\ln x}{x} = \infty \), you can apply L’Hopital’s Rule, as follows

\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{d}{dx} [\ln x]}{\frac{d}{dx} [x]} = \lim_{x \to \infty} \frac{1}{x} = 0
\]

2. (i) Since \( f(x) = ax^2(2-x) \) is a probability density function on the interval \([0, 2]\). So, evaluate the following integral

\[
1 = \int_0^2 ax^2(2-x) \, dx = a \int_0^2 x^2(2-x) \, dx = a \left[ \frac{2}{3} - \frac{1}{4} \right] = a \left[ \frac{16}{3} - 4 \right] = \frac{4a}{3}
\]

Thus, \( a = \frac{3}{4} \).

(ii)

\[
P \left(0 \leq x \leq \frac{1}{2}\right) = \int_0^{1/2} \frac{3}{4} x^2(2-x) \, dx = \frac{3}{4} \int_0^{1/2} x^2(2-x) \, dx = \frac{3}{4} \left[ \frac{2}{3} x^3 - \frac{1}{4} x^4 \right]_0^{1/2} = \frac{13}{256}
\]

3. (i) First you interchange the order of integration so that \( y \) is the outer variable, then \( x \) will have constant bounds of integration given by \( 0 \leq y \leq 1 \). Solving for \( x \) in the equation \( y = \sqrt{x} \) implies that the bounds for \( x \) are \( 0 \leq x \leq y^2 \). Thus

\[
\int_0^1 \int_{\sqrt{y}}^1 \sin \left( \frac{y^3 + 1}{2} \right) \, dy \, dx = \int_0^1 \int_0^y \sin \left( \frac{y^3 + 1}{2} \right) \, dx \, dy = \int_0^1 \sin \left( \frac{y^3 + 1}{2} \right) \, y^2 \, dy
\]
\[ u = \frac{y^3 + 1}{2}, y : 0 \to 1 \Rightarrow du = \frac{3}{2}y^2 \, dy, u : 1/2 \to 1 \]
\[ = \int_{1/2}^{1} \sin(u) \frac{2}{3} \, du = \frac{2}{3} \int_{1/2}^{1} \sin(u) \, du = -\frac{2}{3} \cos(u)|_{1/2}^{1} \]
\[ = -\frac{2}{3} \left[ \cos(1) - \cos(1/2) \right] \]

(ii) By integration by parts, we can written the integral as follows
\[ \int_{0}^{1} x f''(x) \, dx = \left[ x f'(x) \right]_{0}^{1} - \int_{0}^{1} f'(x) \, dx = \left[ x f'(x) \right]_{0}^{1} - \int_{0}^{1} f(x) \, dx = 3f'(3) - f(3) + f(0) \]

By assumption \( f(0) = 5, f(3) = 5, \) and \( f'(3) = 4. \) Thus
\[ \int_{0}^{1} x f''(x) \, dx = 3(4) - 5 + 5 = 12 \]

4. (i) Let \( u = 1 + \cos^2 t, \) \( du = -2 \sin t \cos t \, dt, \) Then the integral
\[ \int \frac{\sin t \cos t}{\sqrt{1 + \cos^2 t}} \, dt = -\frac{1}{2} \int \frac{1}{\sqrt{u}} \, du = -u^{1/2} + C = -\sqrt{1 + \cos^2 t} + C \]

(ii)
\[ \int_{-\infty}^{\infty} \frac{e^x}{(1 + e^x)^3} \, dx = \lim_{a \to -\infty} \int_{a}^{0} \frac{e^x}{(1 + e^x)^3} \, dx + \lim_{b \to \infty} \int_{0}^{b} \frac{e^x}{(1 + e^x)^3} \, dx \]
\[ (u = 1 + e^x, x : a \to 0, x : 0 \to b) \Rightarrow du = e^x \, dx, u : 1 + e^a \to 2, u : 2 \to 1 + e^b) \]
\[ = \lim_{a \to -\infty} \int_{1+e^a}^{2} \frac{1}{u^3} \, du + \lim_{b \to \infty} \int_{2}^{1+e^b} \frac{1}{u^3} \, du \]
\[ = \lim_{a \to -\infty} \left[ -\frac{1}{2u^2} \right]_{1+e^a}^{2} + \lim_{b \to \infty} \left[ -\frac{1}{2u^2} \right]_{2}^{1+e^b} \]
\[ = \lim_{a \to -\infty} \left[ -\frac{1}{8} + \frac{1}{2(1+e^a)^2} \right] + \lim_{b \to \infty} \left[ -\frac{1}{8} + \frac{1}{2(1+e^b)^2} + \frac{1}{8} \right] \]
\[ = -\frac{1}{8} + \lim_{a \to -\infty} \frac{1}{2(1+e^a)^2} - \lim_{b \to \infty} \frac{1}{2(1+e^b)^2} + \frac{1}{8} \]
\[ = -\frac{1}{8} + \frac{1}{2} + \frac{1}{8} = \frac{1}{2} \]

5. (i) \( y' = y(1 - y) \Rightarrow \frac{dy}{dt} = y(1 - y) \Rightarrow \int \frac{1}{y(1 - y)} \, dy = dt \)

Integrate both sides \( \int dt = \int \frac{1}{y(1 - y)} \, dy. \) Then
\[ t + C_1 = \int \frac{1}{y} \, dy + \int \frac{1}{1 - y} \, dy = \ln |y| - \ln |1 - y| = \ln \left| \frac{y}{1 - y} \right| \]

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\[ \Rightarrow \ln C e^t = \ln \left| \frac{y}{1 - y} \right| \Rightarrow C e^t = \frac{y}{1 - y} \Rightarrow C e^t - y C e^t = y \]

\[ \Rightarrow (1 + C e^t) y = C e^t \Rightarrow y(t) = \frac{C e^t}{1 + C e^t} \]

(ii) \[ \frac{1}{3} = y(0) = \frac{C e^{(0)}}{1 + C e^{(0)}} = \frac{C}{1 + C} \Rightarrow C = \frac{1}{2}. \]
Thus the solution for the equation with initial condition is \[ y(t) = \frac{e^t}{2 + e^t} \]

6. (i) When the ball hits the ground the first time, it has traveled a distance of
\[ D_1 = 18 \]
Between the first and second times it hits the ground, it has traveled an additional distance of
\[ D_2 = 18 \left( \frac{7}{10} \right) + 18 \left( \frac{7}{10} \right) = 36 \left( \frac{7}{10} \right) \]
Between the second and third times the ball hits the ground, it has traveled an additional distance of
\[ D_3 = 18 \left( \frac{7}{10} \right) \left( \frac{7}{10} \right) + 18 \left( \frac{7}{10} \right) \left( \frac{7}{10} \right) = 36 \left( \frac{7}{10} \right)^2 \]
Continuing this process, you obtain a total vertical distance traveled of
\[ D = 18 + 36 \left( \frac{7}{10} \right) + 36 \left( \frac{7}{10} \right)^2 + \cdots = -18 + 36 \left( \frac{7}{10} \right) + 36 \left( \frac{7}{10} \right)^2 + \cdots \]
\[ = -18 + \sum_{n=0}^{\infty} 36 \left( \frac{7}{10} \right)^n = -18 + \frac{36}{1 - \left( \frac{7}{10} \right)} = -18 + 120 = 102 \]

(ii) The infinite series
\[ \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2}} = \frac{1}{1^{3/2}} + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots \]
is a \( p \)-series with \( p = \frac{3}{2} \). Because \( p < 1 \), you can conclude that the series divergence.

7. (i) The first derivative of \( f \) is \( f'(x) = 3x^2 \). Thus, the iterative formula for Newton’s Method is
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x^3 - 66}{3x^2} \]
The calculations for two iterations are shown in the table.

<table>
<thead>
<tr>
<th>n</th>
<th>$x_n$</th>
<th>$f(x_n)$</th>
<th>$f'(x_n)$</th>
<th>$f_{n-1}(x_n)$</th>
<th>$x_n - f_{n-1}(x_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>-2</td>
<td>48</td>
<td>0.04167</td>
<td>4.04167</td>
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<td>2</td>
<td>4.04167</td>
<td>0.02107</td>
<td>49.00529</td>
<td>0.00043</td>
<td>4.0412</td>
</tr>
<tr>
<td>3</td>
<td>4.0412</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus, the approximation is $\sqrt[3]{67} = 4.0412$

(ii) Begin by finding several derivatives of $f$ and evaluation each at $c = 64$

\[
g(x) = x^{1/3} \quad g(64) = 4
\]

\[
g'(x) = \frac{1}{3}x^{-2/3} \quad g'(64) = \frac{1}{3}(0.0625) = 0.02083
\]

\[
g''(x) = -\frac{2}{9}x^{-5/3} \quad g''(64) = -\frac{2}{9}(0.00098) = -0.00011
\]

Thus, the two-degree Taylor polynomial is

\[
g(x) = g(64) + g'(64)(x - 64) + \frac{g''(64)(x - 64)^2}{2!} = 4 + 0.02083(x - 64) - 0.00011(x - 64)^2
\]

To evaluate the series when $x = 66$.

\[
g(66) = 4 + 0.02083(66 - 64) - 0.00011(66 - 64)^2 = 4 + 0.02083(2) - 0.00011(4) = 4.04122
\]

8. (i) \[E[X] = 0P(X = 0) + 1P(X = 1) + 2P(X = 2) + 3P(X = 3) + \cdots \]

\[= e^{-2} \sum_{n=0}^{\infty} \frac{2^n}{n!} = 2e^{-2}e^2 = 2\]

(ii)

\[
\mu = \int_{0}^{\infty} x (5e^{-5x}) \, dx = \lim_{a \to \infty} \int_{0}^{a} x (5e^{-5x}) \, dx
\]

\[= \lim_{a \to \infty} \left[ -xe^{-5x} \bigg|_{0}^{a} + \int_{0}^{a} e^{-5x} \, dx \right] = \lim_{a \to \infty} \left[ -ae^{-5a} - \frac{1}{5} e^{-5a} \bigg|_{0}^{a} \right]
\]

\[= \lim_{a \to \infty} \left[ -ae^{-5a} - \frac{1}{5} e^{-5a} + \frac{1}{5} \right] = \lim_{a \to \infty} \frac{-a}{e^{-5a}} - \lim_{a \to \infty} \frac{1}{5e^{5a}} + \frac{1}{5}
\]

\[= - \lim_{a \to \infty} \frac{1}{5e^{5a}} + \frac{1}{5} = \frac{1}{5}
\]

\[
\int_{0}^{\infty} x^2 (5e^{-5x}) \, dx = \lim_{a \to \infty} \int_{0}^{a} x^2 (5e^{-5x}) \, dx
\]
\[
\lim_{a \to \infty} \left[ -x^2 e^{-5x} \bigg|_0^a - \frac{2}{5} x e^{-5x} \bigg|_0^a + \frac{2}{5} \int_0^a e^{-5x} \, dx \right]
\]
\[
= \lim_{a \to \infty} \left[ -a^2 e^{-5a} - \frac{2a}{5e^{5a}} - \frac{2}{5} e^{-5a} \bigg|_0^a \right]
\]
\[
= \lim_{a \to \infty} \frac{-a^2}{e^{5a}} - \lim_{a \to \infty} \frac{2a}{5e^{5a}} - \lim_{a \to \infty} \frac{2}{25e^{5a}} + \frac{2}{25}
\]
\[
= - \lim_{a \to \infty} \frac{2a}{5e^{5a}} - \lim_{a \to \infty} \frac{2}{25e^{5a}} + \frac{2}{25}
\]
\[
= \frac{2}{25}
\]

Thus
\[
\sigma = \sqrt{\frac{2}{25} - \left(\frac{1}{5}\right)^2} = \sqrt{\frac{1}{25}} = \frac{1}{5}
\]

9. Finding several derivatives of \( f \) and evaluation each at \( c = 0 \)

\[
f(x) = \ln \left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) \quad f(0) = 0
\]
\[
f'(x) = \frac{1}{1+x} - \frac{-1}{1-x} = \frac{2}{1-x^2} \quad f'(0) = 2
\]
\[
f''(x) = \frac{0 - (-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2} \quad f''(0) = 0
\]
\[
f^{(3)}(x) = \frac{2 \left(1-x^2\right)^2 - 2x(2(1-x^2)(-2x))}{(1-x^2)^4} = \frac{2 + 6x^2}{(1-x^2)^3} \quad f^{(3)}(0) = 2
\]

Continuing this process, we can see that \( f^{(2n-2)}(0) = 0, f^{(2n-1)}(0) = 2 \),

Thus the Taylor series
\[
f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \frac{f'(0)x^4}{4!} + \cdots
\]
\[
= 2x + \frac{2}{3!} x^3 + \frac{2}{5!} x^5 + \cdots = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{(2n+1)!}
\]

For this power series, \( a_n = \frac{2}{(2n+1)!} \)

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}x^{2(n+1)+1}}{a_n x^{2n+1}} \right| = \lim_{n \to \infty} \left| \frac{2(2n+3)!x^{2n+3}}{(2n+1)!x^{2n+1}} \right| = \lim_{n \to \infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0
\]

So, by the Ratio Test, this series converges for all \( x \) and the radius of convergence is \( \infty \).
10. Let \( T(x, y, z) = 8x^2yz \) and \( g(x, y, z) = x^2 + y^2 + z^2 - 1 \). Then, define a new function \( F(x, y, x, \lambda) \) by

\[
F(x, y, x, \lambda) = T(x, y, z) - \lambda(x, y, z) = 8x^2yz - \lambda(x^2 + y^2 + z^2 - 1)
\]

To find the critical numbers of \( F \), set the partial derivatives of \( F \) with respect to \( x, y, z, \) and \( \lambda \) equal to zero and obtain

\[
F_x(x, y, x, \lambda) = 16xyz - 2\lambda x = 0, \quad F_y(x, y, x, \lambda) = 8x^2z - 2\lambda y = 0,
\]

\[
F_z(x, y, x, \lambda) = 8x^2y - 2\lambda z = 0, \quad F_\lambda(x, y, x, \lambda) = -x^2 - y^2 - z^2 + 1 = 0
\]

Then

\[
\frac{F_x}{F_y} : \frac{2y}{x} = \frac{x}{y}, \quad \frac{F_y}{F_z} : \frac{z}{y} = \frac{y}{z}, \quad \frac{F_z}{F_x} : \frac{2z}{x} = \frac{x}{z}
\]

\[
\Rightarrow x^2 = 2y^2, \quad z^2 = y^2
\]

Substitute this into the equation \( F_\lambda(x, y, x, \lambda) = -x^2 - y^2 - z^2 + 1 = 0 \) and solve \( y \)

\[
0 = F_\lambda(x, y, x, \lambda) = -2y^2 - y^2 - y^2 + 1 \Rightarrow y^2 = \frac{1}{4} = \pm \frac{1}{2}
\]

Using this \( y \)-value, you can conclude that the critical values are

\[
y = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}, \quad z = \pm \frac{1}{2}
\]

\[
x = -\frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}, \quad z = \pm \frac{1}{2}
\]

which implies that the temperature value is

\[
T\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right) = 8(1/2)(1/2)(1/2) = 1, \quad T\left(\pm \frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right) = 8(1/2)(-1/2)(1/2) = -1,
\]

\[
T\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2}\right) = 8(1/2)(1/2)(-1/2) = -1, \quad T\left(\pm \frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{1}{2}\right) = 8(1/2)(1/2)(1/2) = 1,
\]

Thus, the point(s) on the sphere at which the temperature is greatest is

\[
\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right), \quad \left(\pm \frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{1}{2}\right)
\]

the point(s) on the sphere at which the temperature is least is

\[
\left(\pm \frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right), \quad \left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2}\right)
\]