Show all your work. Explanation is required for each problem.

1. [10%] Compute the surface integral \( \iint_S x^2 \, dS \) where \( S \) is the unit sphere \( x^2 + y^2 + z^2 = 1 \).

2. A set \( A \subset \mathbb{R}^n \) is said to be dense in a set \( B \subset \mathbb{R}^n \) if \( A \subset B \) and \( B \subset \text{cl}(A) \) (the closure of \( A \)).
   (i) [6%] If \( A \) is dense in \( \mathbb{R}^n \) and \( U \subset \mathbb{R}^n \) is open, prove that \( A \cap U \) is dense in \( U \).
   (ii) [6%] Give an example of a set \( A \subset \mathbb{R}^n \) and a closed set \( V \subset \mathbb{R}^n \) such that \( A \) is dense in \( \mathbb{R}^n \) but \( A \cap V \) is not dense in \( V \).

3. (i) [6%] Give an example of a bounded function \( f : [0, 1] \to \mathbb{R} \) such that \( |f| \) is Riemann-integrable on \([0, 1]\) but \( f \) is not Riemann-integrable on \([0, 1]\).
   (ii) [6%] Give an example of a function which is bounded and continuous but not uniformly continuous.

4. Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
   
   \[
   f(x, y) := \begin{cases} 
   0, & \text{if } (x, y) = (0, 0); \\
   \frac{x^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0).
   \end{cases}
   \]

   (i) [6%] Prove that \( f \) is continuous.
   (ii) [6%] Prove that \( f \) is not differentiable at \((0, 0)\).

5. For \( n = 1, 2, 3, \ldots \) and \( x \in \mathbb{R} \), we define
   \[
   f_n(x) := \frac{x}{1 + nx^2}.
   \]

   (i) [6%] Prove that the sequence \( \{f_n\} \) converges uniformly on \( \mathbb{R} \) to a differentiable function \( f \).
   (ii) [6%] Prove that \( f' \neq \lim_{n \to \infty} f'_n \) at some point in \( \mathbb{R} \).

6. Define \( \ln(x) := \int_1^x \frac{1}{t} \, dt \) for \( x > 0 \).
   (i) [6%] Prove (from above definition) that \( \ln(ab) = \ln(a) + \ln(b) \) for any \( a, b > 0 \).
   (ii) [6%] Prove that \( \lim_{x \to 0} \ln(x) = -\infty \).
   (iii) [6%] Prove that \( \ln(x) \) has a differentiable inverse function (denoted by \( \exp(x) \)) and prove that \( \frac{d}{dx} \exp(x) = \exp(x) \).

7. Let \( A \) be a subset of \( \mathbb{R} \). We define \( \lambda(A) \in \mathbb{R} \cup \{\infty\} \) as follows. First, if \( A \) is an open interval \((a, b)\), then we define \( \lambda(A) := b - a \). Second, if \( A \) is an open set, we know that \( A \) is a union of countable (including finite) disjoint open intervals: \( \bigcup_{k=1}^{\infty} (a_k, b_k) \) (or \( \bigcup_{k=1}^{n} (a_k, b_k) \)). Then we define \( \lambda(A) := \sum_{k=1}^{\infty} (b_k - a_k) \) (or \( \sum_{k=1}^{n} (b_k - a_k) \)). Finally, if \( A \) is any subset of \( \mathbb{R} \), we define \( \lambda(A) := \inf \{ \lambda(X) \mid A \subset X, X \subset \mathbb{R} \text{ and } X \text{ is open} \} \).

   (i) [6%] Show that \( \lambda([a, b]) = b - a \).
   (ii) [6%] If \( A \subset B \subset \mathbb{R} \), prove that \( \lambda(A) \leq \lambda(B) \).
   (iii) [6%] Suppose we know that \( \lambda(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \lambda(A_k) \) for subsets \( A_1, A_2, A_3, \ldots \) of \( \mathbb{R} \). Compute \( \lambda(Q) \).
   (iv) [6%] Give an example of subsets \( A_1, A_2, A_3, \ldots \) of \( \mathbb{R} \) such that \( A_1 \supset A_2 \supset A_3 \supset \cdots \) and \( \lambda(\lim_{k \to \infty} A_k) \neq \lim_{k \to \infty} \lambda(A_k) \).