1. (10 points) Let \( g_n : I = [0, 1] \to \mathbb{R} \) be defined by \( g_n(x) = \frac{1}{nx+1} \). Determine whether \( g_n \) converge on \( I = [0, 1] \) and, if it converges, determine whether the convergence is uniform.

2. (10 points) Let \( f : I = [a, b] \to \mathbb{R} \) be (Riemann) integrable on \( I \) and assume that \( f \) is continuous at \( c \in (a, b) \). Prove that \( \lim_{r \to 0} \frac{1}{2r} \int_{c-r}^{c+r} f(x)dx = f(c) \).

3. (10 points) Let \( D \) be the rectangle in \( \mathbb{R} \times \mathbb{R} \) given by \( D = \{ (x, t) | a \leq x \leq b, c \leq t \leq d \} \). Let \( f \) and its partial derivative \( f_t \) be continuous functions defined on \( D \), and \( F \) be a function defined on \([c, d]\) given by \( F(t) = \int_a^b f(x, t)dx \). Prove that \( F \) has a derivative on \([c, d]\) and \( F'(t) = \int_a^b f(x, t)dx \).

4. Let \( \sup S \) denotes the supremum (or the least upper bound) of \( S \), and \( \inf S \) denotes the infimum (or the greatest lower bound) of \( S \).

   (a) (6 points) Let \( I = (a, b) \) be an open interval in \( \mathbb{R} \), and let \( f \) and \( g \) be continuous functions defined on \( I \). Prove that the function \( h : I \to \mathbb{R} \) defined by \( h(x) = \sup \{ f(x), g(x) \} \) is continuous on \( I \).

   (b) (6 points) Let \( X \) and \( Y \) be non-empty sets and let \( f : X \times Y \to \mathbb{R} \) have bounded range in \( \mathbb{R} \). Prove that \( \sup_{x, y} f(x, y) \leq \inf_y \inf_x f(x, y) \).

5. Let \( F : \mathbb{R}^5 \to \mathbb{R}^2 \) be defined by \( F(u, v, w, x, y) = (ux + vx + w + x^2,uvw + x + y + 1) \), and note that \( F(2, 1, 0, -1, 0) = (0, 0) \).

   (a) (6 points) Show that we can solve \( F(u, v, w, x, y) = (0, 0) \) for \((x, y)\) in terms of \((u, v, w)\) near \((2, 1, 0)\).

   (b) (6 points) If \((x, y) = \Phi(u, v, w)\) is the solution of the preceding part, show that \( D\Phi(2, 1, 0) \) is given by the matrix \( \begin{pmatrix} -1 & 2 & 0 & -1 & 1 \\ 1 & 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix} \).

6. (10 points) Define a sequence of real numbers \( (x_n) \) by \( x_0 = 1 \), and \( x_{n+1} = \frac{1}{2 + x_n} \), for \( n \geq 0 \). Show that \( (x_n) \) converges and compute its limit. [Hint: Use the contraction principle.]

7. Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function with \( \lim_{x \to \infty} f(x) = 0 \).

   (a) (10 points) Show that there exists a sequence \( x_n \to \infty \) with \( \lim_{n \to \infty} f(x_n) = 0 \).

   (b) (6 points) Show that it is not necessarily true that \( f'(x) \) is bounded.

8. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be differentiable for each \( n \), so that \( |f_n'(x)| \leq 1 \), for all \( x \in \mathbb{R}, n = 1, 2, \ldots \).

   (a) (6 points) Prove that the set \( \{ f_n \} \) is uniformly equicontinuous on \( \mathbb{R} \). [Hint: A set \( \mathcal{F} \) of functions on \( K \) to \( \mathbb{R}^d \) is said to be uniformly equicontinuous on \( K \) if, for each \( \varepsilon > 0 \) there is a \( \delta(\varepsilon) > 0 \) such that if \( x, y \in K \) and \( |x - y| < \delta(\varepsilon) \) and \( f \in \mathcal{F} \), then \( ||f(x) - f(y)|| < \varepsilon. \)]

   (b) (6 points) For each \( n \), let \( \tilde{f}_n(x) = f_n(x) - f_n(0) \). Prove that \( \{ \tilde{f}_n \} \) is uniformly bounded on any closed interval \([a, b] \subseteq \mathbb{R} \).

   (c) (8 points) Suppose that \( g : \mathbb{R} \to \mathbb{R} \) is such that for each \( x \in \mathbb{R} \), \( \lim_{n \to \infty} f_n(x) = g(x) \). Prove that \( g \) is continuous.