1. [10%] Prove that $\sqrt{2} + \sqrt{3}$ is a root of $x^4 - 10x^2 + 1 = 0$, and hence establish that it is irrational.

2. [10%] Suppose that $p$ is an odd rational prime. Show that

$\left( \frac{3}{p} \right) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{3}; \\
-1, & \text{if } p \equiv 2 \pmod{3}. \end{cases}$

where $\left( \frac{a}{p} \right)$ denotes the Legendre symbol.

3. [10%] Recall that the function $\mu$ is defined by

$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\
0, & \text{if } a^2 | n \text{ for some rational integer } a > 1; \\
(-1)^r, & \text{if } n = p_1 p_2 \cdots p_r \text{ where } p_1, \ldots, p_r \text{ are distinct rational primes}. \end{cases}$

Find a positive rational integer $n$ such that $\mu(n) + \mu(n + 1) + \mu(n + 2) = 3$.

4. [10%] Show that the primitive solutions of $x^2 + y^2 = z^2$ with $y$ even are $x = r^2 - s^2$, $y = 2rs$, $z = r^2 + s^2$, where $r$ and $s$ are arbitrary integers of opposite parity with $r > s > 0$ and $\gcd(r, s) = 1$.

5. [10%] Let $\{a_i\}$ and $\{b_i\}$ be increasing sequences of real numbers. We say that $a_i$ is asymptotic to $b_i$, write $a_i \sim b_i$, if and only if $\lim_{i \to \infty} a_i/b_i = 1$. Prove that $a_i \sim b_i$ implies $\ln a_i \sim \ln b_i$, but the converse is not true.

6. [10%] Let $a_0, a_1, a_2, \ldots$ be a sequence of rational integers, all positive except perhaps $a_0$. Recall that we define the sequence of rational integers $\{k_n\}$ by $k_{-2} = 1$, $k_{-1} = 0$ and $k_i = a_i k_{i-1} + k_{i-2}$ for $i \geq 0$. Show that $k_n/k_{n-1} = \langle a_n, a_{n-1}, \ldots, a_2, a_1 \rangle$ for $n \geq 1$.

7. [10%] Let $k$ and $r$ be positive rational integers with $k > 1$ and $r > 1$. Prove that there is a rational prime whose digital representation to base $r$ has exactly $k$ digits.

8. [10%] Show that 3 is a prime in $\mathbb{Q}(\sqrt{-1})$, but not a prime in $\mathbb{Q}(\sqrt{6})$.

9. [10%] The rational prime 13 can be factored in two ways in $\mathbb{Q}(\sqrt{-3})$,

$13 = \frac{7 + \sqrt{-3}}{2} \cdot \frac{7 + \sqrt{-3}}{2} = (1 + 2\sqrt{-3})(1 - 2\sqrt{-3})$.

Explain why this does not conflict with the fact that $\mathbb{Q}(\sqrt{-3})$ has the unique factorization property.

10. [10%] Let $p(n)$ denote the number of partitions of $n$. Compute $p(9)$. 