PhD Qualify Exam in Numerical Analysis
March 11, 2015

1. (Average)

   (a) (5 %) Show that the nontrivial fixed point of the equation $4x = x^3, x \in [0, \infty)$ is unstable, that is, you cannot find the nontrivial fixed point by using the fixed point iteration $4x_{k+1} = x_k^3$ with arbitrary $x_0 \neq 2 \in [0, \infty)$.

   (b) (10 %) Please develop a new fixed iteration method such that the nontrivial fixed point of the equation $4x = x^3, x \in [0, \infty)$ is stable.

2. (Average)

   Consider the heat equation with Dirichlet boundary conditions:

$$
\begin{aligned}
u_t &= u_{xx} \quad \text{for } 0 < x < 1,\ 0 < t < T, \\
u(0,t) &= 0, \quad \text{for } 0 < t < T, \\
u(1,t) &= 0, \quad \text{for } 0 < t < T.
\end{aligned}
$$

   We attempt to solve the problem using a finite difference scheme on a discrete with grid points (x_i, t_n) where $x_i = ih$, $t_n = nk$. Here $h = 1/(m+1)$ is the mesh spacing on the x-axis and k is the time step. Let $U_i^n \approx u(x_i, t_n)$ represent the numerical solution at grid point (x_i, t_n).

   The finite difference is

$$
\begin{aligned}
U_i^{n+1} &= U_i^n + \frac{k}{h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n),. \quad \text{for i=1,}\ldots\text{, m,} \\
U_0^n &= 0, \quad U_{m+1}^n = 0.
\end{aligned}
$$

   (a) (10 %) Determine the order of accuracy of this method (in both space and time).

   (b) (10 %) Suppose we take $k = \lambda h^2$ for some fixed $\lambda > 0$ and refine the grid. Show that this method is stable for $0 < \lambda \leq 1/2$ and hence convergent.

   (c) (5 %) Based on the CFL Condition, a numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as k and h go to zero.

   What is the domain of dependence of the heat equation? Does this explicit scheme satisfy the CFL condition?

3. (Easy) (10 %) Show that if $u(x)$ is a function that interpolates $f(x)$ at $x_0, x_1, \ldots, x_{n-1}$ and $v(x)$ is a function that interpolates $f(x)$ at $x_1, x_2, \ldots, x_n$ then the function $w(x)$ given by

$$w(x) = \frac{(x_n - x)u(x) + (x - x_0)v(x)}{x_n - x_0}$$

interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$.

4. (Average) A sequence $\{p_n\}$ is said to be **superlinearly convergent** to p if

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0.$$

   (a) (10 %) Show that if $p_n \to p$ of order α for $\alpha > 1$, then the sequence $\{p_n\}$ is certainly superlinearly convergent to p.

   (b) (5 %) Show that $\{p_n = \frac{1}{n^n}\}$ is superlinearly convergent to 0, but does not converge to 0 of any order α for $\alpha > 1$.

5. (Average) Let $A = \begin{bmatrix} a & a - \varepsilon \\ 2(a + \varepsilon) & 2a \end{bmatrix}$ where $a \approx O(1)$ and ε is sufficiently small.

   (a) (5 %) Find A^{-1}.

   (b) (10 %) Choose b, δb, x and δx such that

$$Ax = b, \qquad A(x + \delta x) = b + \delta b,$$

   and $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$ is small, but $\frac{\|\delta x\|_\infty}{\|x\|_\infty}$ is large.

   (c) (10 %) Choose b, δb, x and δx such that

$$Ax = b, \qquad A(x + \delta x) = b + \delta b,$$

   and $\frac{\|\delta x\|_\infty}{\|x\|_\infty}$ is small, but $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$ is large.

6. (Easy) (10 %) Show that there is a unique quadratic polynomial $p_2(x)$ satisfying the conditions

$$p_2(0) = a_0, \quad p_2(1) = a_1 \text{ and } \int_0^1 p_2(x)dx = \bar{a}$$

with given a_0, a_1 and $\bar{a} \in \mathbb{R}$.