PhD Qualify Exam: General Analysis

2nd October, 2015

E: Easy; M: Moderate; D: Difficult.

Problem A.(10 × 4 points) True or false. Explain it.

(1) (E) Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function and for all $a \in \mathbb{R}$, $\{x : f(x) = a\}$ is a measurable set, then f is a measurable function.

(2) (E, Oct. 2014) Let $f_k, f : \mathbb{R} \to \mathbb{R}$ be real valued functions and f_k converge to f in $L^2(\mathbb{R})$, then f_k converges to f in measure.

(3) (E) Let f be a function of bounded variation, then f is an absolutely continuous function.

(4) (M) Let $a = \{a_k\}_{k=1}^{\infty} \in l^p$ for some $p < \infty$, then

$$\lim_{p\to\infty} \|a\|_p = \|a\|_\infty .$$

Problem B. (15 × 4 points) Prove the following statements:

(5) (E) Let $\{g_n\}$ be an integral function, $g_n \to g$ a.e. and $|f_n| \le g_n$, $f_n \to f$ a.e.. If

$$\int g dx = \lim_{n\to\infty} \int g_n dx ,$$

then

$$\int f dx = \lim_{n\to\infty} \int f_n dx .$$

(6) (E, Sep. 2007) Let g be a non-negative measurable function on $[0, 1]$, then

$$\log \int g(t)dt \ge \int \log g(t)dt .$$

(7) (M) Let $1 \le p < \infty$ and g be an integral function on $[0, 1]$, suppose that there exists $M > 0$ such that

$$\left| \int_0^1 f g dx \right| \le M\|f\|_p$$

for all bounded measurable function f, then $g \in L^q$ and $\|g\|_q \le M$, where $1/p + 1/q = 1$.

(8) (E, Oct. 2014) Let $f \in L^2(0,\infty)$,

$$F(x) = \frac{1}{x}\int_0^x f(t)dt\,,$$

where $0 < x < \infty$, then

$$\|F\|_{L^2} \le 2\|f\|_{L^2}\,.$$