Qualifying Exam, Oct 2015
Differential Geometry

(**E**: Easy, **M**: Moderate, **D**: Difficult)

1. Let $\omega = a(x,y)dx + b(x,y)dy$ be a smooth 1-form on $\mathbb{R}^2$ such that $d\omega = 0$.
   (a) (**E**, 10%) Find the relation between $\frac{\partial a}{\partial y}(x,y)$ and $\frac{\partial b}{\partial x}(x,y)$.
   (b) (**M**, 10%) Show that $\omega = df$ where
   $$f(x,y) = \int_0^1 \{x\, a(tx,ty) + y\, b(tx,ty)\}\, dt.$$
   [Hint: $\frac{d}{dt}(ta(tx,ty)) = a(tx,ty) + tx\frac{\partial a}{\partial x}(tx,ty) + ty\frac{\partial a}{\partial y}(tx,ty)$.]

2. Let $GL(n,\mathbb{R})$ be the set of all invertible $n \times n$ real matrices, and let $SL(n,\mathbb{R})$ be the subset of $GL(n,\mathbb{R})$ consisting of matrices of determinant 1. We view $GL(n,\mathbb{R})$ and $SL(n,\mathbb{R})$ as subspaces of the Euclidean space $\mathbb{R}^{n^2}$.
   (a) (**E**, 10%) Show that $GL(n,\mathbb{R})$ is a smooth manifold.
   (b) (**M**, 10%) Show that $SL(n,\mathbb{R})$ is a smooth submanifold of $GL(n,\mathbb{R})$. What is the **dimension** of $SL(n,\mathbb{R})$?

3. Let M and N be smooth manifolds, and let $f : M \to N$ be a smooth map.
   (a) (**E**, 10%) Show that if f is a submersion, then f is an **open map.**
   (b) (**E**, 10%) Show that if M and N have the same dimension and f is an immersion, then f is a **local diffeomorphism.**

4. Let M be a smooth manifold. A critical point of $f \in C^\infty(M)$ is a point $p \in M$ such that $df_p = 0$. Let T_pM be the tangent space of M at p.
   (a) (**E**, 10%) Suppose p is a critical point of f, then we define $H\colon T_pM \times T_pM \to \mathbb{R}$ by
   $$H(v,w) = XYf(p),$$
   where X, Y are smooth vector fields on M and $X_p = v$, $Y_p = w$. Show that H is **well-defined, bilinear and symmetric.**
   (b) (**M**, 10%) Let $\gamma : \mathbb{R} \to M$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = v$. Show that
   $$H(v,v) = \frac{d^2(f \circ \gamma)}{dt^2}(0).$$

5. (**E**, 10%) Let G be a Lie group. Show that the tangent bundle of G is **trivial.**

6. (**E**, 10%) Let M be a smooth compact manifold. Prove that there exists a **Riemannian metric** on M.