Qualifying Exam
Differential Geometry

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- Make sure ALL electronic devices are OFF.
- You have 2 hours.
- Show ALL the work.
- Good Luck!

All manifolds are assumed to be smooth and finite dimensional.

1. The space of n matrices, $Mat(n, \mathbb{R})$, can be identified with $\mathbb{R}^{n^2}$ in the natural way. Define $GL(n, \mathbb{R})$ to be the set of *invertible* n-matrices, and $O(n, \mathbb{R})$ to be the set of *orthogonal matrices*. Precisely,

$$GL(n, \mathbb{R}) := \{A \in Mat(n, \mathbb{R}) \mid det A \neq 0\},$$

$$O(n, \mathbb{R}) := \{A \in Mat(n, \mathbb{R}) \mid AA^T = Id\}.$$

   (a) (5 points) (Easy) Prove that $GL(n, \mathbb{R})$ is a smooth manifold. What is its dimension?

   (b) (10 points) (Medium) Prove that $O(2, \mathbb{R})$ is a smooth manifold. What is its dimension?

2. Let ω be a smooth 1 form on a manifold M.

   (a) (5 points) (Very Easy) State the definition for ω to be a *closed* 1 form.

   (b) (10 points) (Medium) Prove that if ω is closed and M is simply connected, then given any two smooth curves $\gamma_1, \gamma_2 : [0, 1] \to M$ starting and ending at the same points,

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

   (c) (5 points) (Medium) If M is not simply connected, give an example of closed 1-form ω on some manifold M that fails to satisfy the conclusion of part (b). (Hint: consider an open subset of $\mathbb{R}^2$).

3. Given a smooth function f and p-form ω on a manifold M,

   (a) (5 points) (Easy) Prove, using local coordinates, that

$$d(f d\omega) = df \wedge d\omega.$$

   (b) (5 points) (Easy)Use the previous part to show that for 1-form ω and vector fields X, Y, we have

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

   Here, $[X, Y]$ is the *Lie bracket* of vector fields.

4. Let M be a connected and compact manifold, N a manifold in general, and $F : M \to N$ is smooth.

   (a) (5 points) (Easy) Prove that F is a closed map.

   (b) (10 points) (Medium) Prove that if F is a *submersion*, then F is an open map.

   (c) (5 points) (Easy) Prove that if F is a submersion *and* N is *non-compact*, then N must be *disconnected*.

5. Given $f : \mathbb{R}^n \to \mathbb{R}$ smooth. Define its *gradient vector field* by

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

and its *Laplacian* by

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

(a) (5 points) (Easy) For $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$, find the flow of ∇f. (Recall that for a vector field X on M, its flow is given by $\theta : M \times J \to M$ so that $\forall t \in J, \frac{\partial}{\partial t}|_{t=0}\theta(p, t) = X_p$.

(b) (5 points) (Easy) Recall that for any tensor field θ and vector field X on M, the *Lie derivative* $\mathcal{L}_X\theta$ is defined in some appropriate way. For a smooth function (ie a 0 tensor) $h : \mathbb{R}^n \to \mathbb{R}$ and vector field $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x_i}$, write down the coordinate representation of $\mathcal{L}_X(h)$.

(c) (10 points) (Medium) For any smooth function h, differential forms ω and η, the followings are true

- $\mathcal{L}_X(dh) = d(\mathcal{L}_X h)$.
- $\mathcal{L}_X(\omega \wedge \eta) = (\mathcal{L}_X\omega) \wedge \eta + \omega \wedge (\mathcal{L}_X\eta)$

Use these to prove that for $f : \mathbb{R}^3 \to \mathbb{R}$,

$$\Delta f = 0 \Leftrightarrow \mathcal{L}_{\nabla f}(dx \wedge dy \wedge dz) = 0.$$

6. Consider inclusion

$$\iota : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}.$$

Let $(x_1, \ldots, x_n)$ be local coordinate of the upper hemisphere of $\mathbb{S}^n$ (ie. $x_n > 0$) and $(u_1, \ldots, u_{n+1})$ be the global coordinates of $\mathbb{R}^{n+1}$.

(a) (5 points) (Easy) Write out ι in the coordinates given above.

(b) (10 points) (Easy) Let $g = \sum_{i=1}^{n+1} du_i^2$ be the Euclidean metric of $\mathbb{R}^{n+1}$. Write out the coordinate representation of ι^*g on the upper hemisphere.