Numerical Analysis, PhD Qualifying Exam, 2020-09

1. (10%) Use the extended Newton divided difference method to obtain a polynomial P of degree at most 4 that takes these values:

x	0	1	2
$P(x)$	2	-4	44
$P'(x)$	-9	4	

2. (10%) Show that the Runge–Kutta method of two functional evaluations

$$k_1 = hf(t_i, w_i), \quad k_2 = hf(t_i+\alpha h, w_i+\beta k_1), \quad w_{i+1} = w_i+[a_1 k_1 + a_2 k_2]$$

is of second order if

$$a_1 + a_2 = 1, \quad \alpha = \beta = \frac{1}{2a_2}$$

3. (Neville's formula) Let f be defined at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$. Show that

$$P_{0,1,\ldots,n}(x) = \frac{(x-x_0)P_{1,2,\ldots,n}(x) - (x-x_n)P_{0,1,\ldots,n-1}(x)}{x_n - x_0},$$

where a subscripted P denotes the Lagrange polynomial that agrees with f at the indicated nodes. (10%)

4. Consider the initial value problem

$$\text{(I.V.P.)} \begin{cases} y' = f(t,y), & a \leq t \leq b, \\ y(a) = \alpha. \end{cases}$$

(a) Show that

$$y'(t_i) = \frac{-3y(t_i) + 4y(t_{i+1}) - y(t_{i+2})}{2h} + \frac{h^2}{3}y'''(\xi_i),$$

where $t_i = a + ih$, for a sufficient small h and for some ξ_i with $t_i \leq \xi_i \leq t_{i+2}$. (5%)

(b) Part (a) suggests the difference method

$$w_{i+2} = 4w_{i+1} - 3w_i - 2hf(t_i, w_i), \quad \text{for } i = 0, 1, \ldots, n-2.$$

Analyze this method for consistency, stability and convergence. (15%)

5. Consider the square linear system $Ax = b$. Let D be diagonal consisting of the diagonals of A. The parametric Jacobi method, called the relaxation of Jacobi iteration (JOR), is expressed by

$$x_{k+1} = x_k - \omega D^{-1}(Ax_k - b).$$

Show that if Jacobi iteration converges then JOR converges for $0 < \omega \leq 1$. (10%)

6. For any $x_0 \in [0, 2\pi]$, the sequence $\{x_n\}$ is defined by

$$x_n = \pi + 0.5\sin(x_{n-1}/2), \ n \geq 1.$$

The sequence $\{x_n\}$ converges or diverges. Why? (15%)

7. (10%) Show that the vector x_* is a solution to the positive definite linear system $Ax = b$ if and only if x_* minimizes

$$\varphi(x) = x^T Ax - 2x^T b.$$

8. (15%) Consider the Poisson problem with Dirichlet boundary conditions:

$$\begin{aligned} u_{xx} + u_{yy} &= f(x,y) \quad \text{for } 0 < x, y < 1, \\ u(0,y) &= \alpha_0(y), \quad u(1,y) = \alpha_1(y), \\ u(x,0) &= \beta_0(x), \quad u(x,1) = \beta_1(x), \end{aligned}$$

We attempt to compute a grid function consisting of values $U_{0,0}$, $U_{1,0}$, $\ldots$, $U_{m+1,m}$, $U_{m+1,m+1}$ where $U_{i,j}$ is our approximation to the solution $u(x_i, y_j)$. Here $x_i = ih$, $y_j = jh$ and $h = 1/(m+1) = \Delta x = \Delta y$. Solve this problem with a centered difference scheme,

$$\frac{1}{h^2}(U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j}) = f_{i,j} = f(x_i, y_j) \quad \text{for } i, j = 1, 2, \ldots, m,$$

and write the equations in the form $AU = F$. Show that $||A^{-1}||_2$ is uniformly bounded as $h \to 0$ and the numerical scheme is stable in the **2-norm**.