Numerical Analysis, Qualifying Exam.   2022-10

1. (15%易) Show that the polynomial interpolating the following data has degree 3.

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

2. (20%中) Let $I \subset \mathbb{R}$ be an interval containing t_0. Assume that f is uniformly Lipschitz continuous with respect to y. Consider the following initial value problem (IVP):

$$\begin{cases} y'(t) = f\big(t, y(t)\big), & t \in I \\ y(t_0) = y_{0.} \end{cases}$$

Let u_j be the approximation at the node t_j to the exact solution $y_j = y(t_j)$ and let f_j denote the value $f(t_j, y_j)$ and set $u_0 = y_0$.

(a) Find the conditions on the coefficients of the following general 2-stage explicit Runge-Kutta method for (IVP) so that the method is of second-order accuracy:

$$u_{n+1} = u_n + hb_1 f_n + hb_2 f(t_n + hc_2, u_n + hc_2 f_n)$$

(b) From part (a), a second-order Runge-Kutta method can be obtained by taking $b_1 = b_2 = \frac{1}{2}$ and $c_2 = 1$. Find the region of absolute stability of the method.

3. (15%中) Consider the linear system $Ax = b$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$. Let D be a diagonal matrix consisting of the diagonal part of A. The parametric Jacobi method, called the relaxation of Jacobi iteration (JOR), is expressed by

$$x_{k+1} = x_k - \omega D^{-1}(Ax_k - b).$$

Show that if Jacobi iteration converges, then JOR converges for $0 < \omega \leq 1$.

4. (10%易) For any $x_0 \in [0, 2\pi]$, the sequence $\{x_n\}$ is defined by

$$x_n = \pi + 0.5 \cdot \sin\left(\frac{x_{n-1}}{2}\right), \qquad n = 1,2,\cdots$$

The sequence $\{x_n\}$ converges or diverges? Why?

5. (10%易) A forward difference formula for $f'(x_0)$ can be expressed by

$$f'(x_0) = \frac{1}{h}[f(x_0 + h) - f(x_0)] - \frac{h}{2} f''(x_0) - \frac{h^2}{6} f'''(x_0) + O(h^3).$$

Use extrapolation to derive an $O(h^3)$ formula for approximating $f'(x_0)$.

6. (10%易) Show that the vector x_* solves a symmetric positive definite linear system $Ax = b$ if and only if x_* minimizes

$$\varphi(x) = x^T Ax - 2x^T b.$$

7. (20%中) Consider a two-point boundary value problem (BVP):

$$\begin{cases} -\alpha u''(x) + \beta u'(x) + u(x) = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

where α, β are two positive constants and $f(x)$ is a given smooth function.

(a) Derive the weak formulation of (BVP) on the Hilbert space $H_0^1(0,1)$ and show that the weakly problem has a unique solution $u \in H_0^1(0,1)$.

(b) Prove that the piecewise linear finite element solution u_h of (BVP) satisfies the following error estimation

$$\|u - u_h\|_{H_0^1(0,1)} \le Ch^1 \|u\|_{H^2(0,1)}$$

provided $u_h \in H_0^1(0,1) \cap H^2(0,1)$.