PhD Qualify Exam
General Analysis

March 2024

E: Easy, M: Moderate, D: Difficult

1. (M, 15 points) (September 2005) Let $\{f_n\}$ be a sequence of functions in $L^p[0,1]$, $1 < p < \infty$, which converge almost everywhere to a function f in $L^p[0,1]$, and suppose that there is a constant M such that $\|f_n\|_p \le M$ for all n. For each function g in $L^q[0,1]$ and $\frac{1}{p}+\frac{1}{q} = 1$, show that
$$\int_0^1 fg = \lim_{n\to\infty} \int_0^1 f_n g.$$

2. (E. 10 points) (September 2008) Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \cos(xt)\, dx$.

3. (E. 10 points)(September 2011) Prove the following form of Jensen's inequality: Let g be a nonnegative measurable function on $[0,1]$ and $\int \log\big(g(t)\big)\, dt$ is defined. Show that
$$\exp\Big(\int_0^1 \log\big(g(t)\big)\, dt\Big) \le \int_0^1 g(t)\, dt.$$

4. (M. 20 points) (March 2012) Prove or disprove
   (a) If f is an increasing continuous function with $f'(x) = 0$ a.e, then f is a constant function.
   (b) If f is an absolutely continuous function with $f'(x) = 0$ a.e., then f is a constant function.

5. (M. 15 points) Let $f \in L^1\big([0,1]\big)$ and let $1 < p < \infty$. Prove that $f \in L^p\big([0,1]\big)$ if and only if
$$\sup_{\{I_j\}} \sum_j |I_j| \left(\frac{1}{|I_j|}\int_{I_j} |f|\right)^p < \infty$$
where the supremum is taken over all finite partitions of $[0,1]$ into intervals $\{I_j\}$.

6. (E. 10 points) For $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^n$, let
$$F = \big\{x \in E \;\big|\; \text{there is } \{x_k\}_{k=1}^{\infty} \subset E\backslash\{x\} \text{ with } x_k \to x \text{ and } f(x_k) \to f(x)\big\}.$$
Prove that $E\backslash F$ is at most countable.

7. (E. 20 points) Let
$$f * g(x) := \int_{-\infty}^{\infty} f(y)g(x-y)\, dy$$
denote the convolution of f and g.
   (a) Let $f, g \in L^(\mathbb{R})$ be two square-integrable functions on (R) (with the usual Lebesgue measure). Show that the convolution $f * g$ is a bounded continuous function on $\mathbb{R}$.
   (b) Instead let $h \in L^1(\mathbb{R})$ be fixed. Show that $A(f) = f * h$ is a bounded operator $L^1(\mathbb{R}) \to L^1(\mathbb{R})$.