Ph.D. QUALIFYING EXAM, DIFFERENTIAL GEOMETRY

(**E: easy; M: moderate; D: difficult**)

1. **(E. 10 points)** Let M, N be two smooth manifolds and $f : M \to N$ be an immersion. Suppose that $\dim(M) = \dim(N)$. Prove that f is a local diffeomorphism.

2. **(30 points)** Recall that $Gl(n;\mathbb{R})$ is the collection of invertible n by n real-valued matrices equipped with the induced topology of $\mathbb{R}^{n^2}$, and $O(n;\mathbb{R})$ is the collection of orthogonal matrices equipped with the induced topology of $\mathbb{R}^{n^2}$.
   a. **(M. 20 points)** Prove that $Gl(n;\mathbb{R})$ and $O(n;\mathbb{R})$ are smooth submanifolds and compute the dimension.
   b. **(M. 10 points)** Find $T_e(O(n;\mathbb{R}))$ where e is the identity matrix.

3. **(E. 20 points)** Prove that $H^1_{dR}(S^1 \times I)$ is non-trivial.

4. **(20 points)** Prove the following two theorems.
   a. **(M. 10 points)** Let M be a compact manifold. Then every smooth vector bundle $\pi : E \to M$ is a subbundle of a trivial bundle.
   b. **(D. 10 points)** Prove that every smooth vector bundle $\pi : E \to M$ is a pull-back of a tautological bundle of Grassmannian $Gr(m;n)$, i.e., there exists a smooth map $f : M \to Gr(m;n)$ such that $E \simeq f^*(F)$ where F is the tautological bundle of $Gr(m;n)$.

5. **(20 points)** Let (M, g) be a Riemannian manifold.
   a. **(M. 10 points)** Write down the definition of Levi-Civita connection (Denote by ∇).
   b. **(D. 10 points)** Let $\{e_i\}$ be a local frame of the tangent bundle induced by a coordinate chart.Write down the definition of Christoffel symbols Γ^k_{ij} and prove that

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k.$$

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