Algebra Qualifying Examination, September 2002

Answer all the problems and show all your works.

1. (15%) Let $G$ be a nonabelian group of order 6. Show that $G$ is isomorphic to $S_3$, the symmetry group of degree 3.

2. (15%) Let $G$ be a group of order 56. Suppose that $G$ has no element of order 14. Show that the Sylow 2-subgroup of $G$ is normal in $G$.

3. (20%) Let $G$ be a group of order 231. Show that the Sylow 11-subgroup of $G$ is in the center of $G$.

4. (15%) Let $R$ be a commutative ring with identity and

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \in R[x].$$

Show that $f(x)$ is a unit in $R[x]$ if and only if $a_0$ is a unit in $R$ and $a_1, \ldots, a_n$ are nilpotent elements in $R$.

5. (10%) Let $R$ be an integral domain and $a, b \in R$. Suppose $a^m = b^m$ and $a^n = b^n$, where $m, n$ are positive integers and $(m, n) = 1$. Prove that $a = b$.

6. (10%) An integral domain $D$ is called a Euclidean domain if there is a function $d: D \setminus \{0\} \to \mathbb{Z}^+$ such that

(1) $d(a) \leq d(ab)$ for any $a, b \in D \setminus \{0\}$

(2) for any $a \in D$ and $b \neq 0$, there are $q, r \in D$ such that $a = qb + r$, where $d(r) < d(b)$ or $r = 0$.

Show that $d(a) = d(e)$ if and only if $a$ is a unit.

7. (15%) (i) Show that a finite extension $E$ of $F$ is also an algebraic extension of $F$.

(ii) Let $K$ be a field and $E$ an extension of $K$. Suppose $u, v \in E$ are roots of an irreducible polynomial $f(x) \in K[x]$. Show that there is a unique field isomorphism $\sigma: K(u) \to K(v)$ such that $\sigma|K = id_K$ and $\sigma(u) = v$.

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