1. (15%) Prove that the set $[0,1]$ is not countable by measure theory. Can you prove this fact by Cantor's diagonal argument?

2. (15%) Let $f$ be a real-valued, measurable function on $\mathbb{R}$ that satisfies the equation

$$f(x + y) = f(x) + f(y)$$

for all $x, y$ in $\mathbb{R}$. Prove that $f(x) = A$, for some number $A$. (Hint: Prove this when $f$ is continuous by examining $f$ on the rationals.)

3. (15%) Show that the function $\frac{\sin x}{x}$ is Riemann integrable on $(-\infty, \infty)$ but that its Lebesgue integral does not exist.

4. (10%) If $f \in L(0, 1)$, show that $x^k f(x) \in L(0, 1)$ for $k = 1, 2, \ldots$ and

$$\int_0^1 x^kf(x)dx \rightarrow 0$$

5. (15%) Find the limit

$$\lim_{n \to \infty} \int_0^n \left(1 + \frac{x^2}{n}ight)^{-2} dx$$

You need to figure out the dominating function.

6. (15%) Let $p > 0$ and $f \in L^p(\mu)$ where $f \geq 0$, and let $f_n = \min(f, n)$. Show that $f_n \in L^p(\mu)$ and $\lim_{n \to \infty} \|f - f_n\|_p = 0$

7. (15%) Let $f(x, y) = \frac{xy}{(x^2 + y^2)^2}$, $(x, y) \in [-1, 1] \times [-1, 1]$ defining $f(0,0) = 0$. Show that the iterated integrals of $f$ over the square are equal

$$\int_{-1}^1 \int_{-1}^1 f(x, y)dxdy = \int_{-1}^1 \int_{-1}^1 f(x, y)dydx = ??$$

Is $f$ integrable?