Answer all the problems and show all your works.

1. (20%) Let $G$ be a group of order 56 with no element of order 14. Prove that
   (i) the Sylow 7-subgroups of $G$ are not normal in $G$, and
   (ii) the Sylow 2-subgroup of $G$ is normal in $G$ and is isomorphic to $Z_2 \times Z_2 \times Z_2$? where $Z_2$ is a group of order 2.

2. (15%) Let $p$ be a prime.
   (i) Show that every group of order $p^2$ is abelian.
   (ii) Suppose that $G$ is a non-abelian group of order $p^3$. Show that the center of $G$ is non-trivial.

3. (10%) (i) Show that every group can be embedded into a symmetric group $S_n$ for some $n$.
   (ii) Show that every group can be embedded into an alternating group $A_n$ for some $n$.

4. (10%) Let $D$ be a principal ideal domain. Show that $I$ is prime ideal in $D$ if and only if it is a maximal ideal.

5. (10%) Let $G$ be a group. Show that $\text{End} G$ is a ring if and only if $G$ is abelian, where $\text{End} G$ is the set of all homomorphisms from $G$ to $G$ and the addition and the multiplication on $\text{End} G$ are defined as follows:
   
   $$(f + g)(a) = f(a)g(a), \quad f \cdot g(a) = f(g(a)),$$
   
   for any $f, g \in \text{End} G$ and $a \in G$

6. (10%) Let $F$ be a finite field. Show that the order of $F$ is a power of a prime.

7. (10%) Let $Q$ be the field of all rational numbers. Show that
   
   $$Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3}).$$

8. (15%) Let $F$ be a field and $A$, $B$ and $C$ $F$-vector spaces. Show that
   
   $$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$
   
   as $F$-vector spaces.

THE END