Qualified Examination: Functional Analysis

Feb 21.2003


1. (15) (M) Prove that the space $L^1(\mathbb{R})$ is not reflexive.

2. (20) (E) Consider the operator $A$ defined by

$$Af(x)=\int K(x,y)f(y)dy \quad \text{(*)}$$

from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq \infty$.

(a) Prove that if $K(x,y) \in L^p(\mathbb{R}^n \times \mathbb{R}^n)$, then $A$ is a bounded operator.
(b) Let $X, Y$ be bounded closed sets in $\mathbb{R}^n$. Denote by $\mu$ the Lebesgue measure. Prove that if $K(x,y)$ is continuous on $X \times Y$, then the operator $A$ defined by (*) is a compact operator from $L^p(Y, \mu)$ into $L^q(X, \mu)$.

3. (15) (M) Let $Y$ be a finite dimensional linear subspace of a normed space $X$. Show that $Y$ must be closed.

4. (20) (E) Let $X$ be a normed linear space, and let $X^*$ be its dual with the norm $\|X^*\|=\sup\{|f(x)| : f \in X^*, x \in X, \|x\| \leq 1, X \}$.

(a) Prove that $X^*$ is a Banach space.
(b) Prove that for each $x \in X$, the mapping $f \mapsto f(x)$ is a bounded linear functional on $X^*$ with norm $\|x\|$.

5. (15) (E) Let $H$ be a separable Hilbert space, and let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of $H$. $T : H \to H$ is a bounded operator with $\sum_{n=1}^\infty \|Te_n\|^2 < \infty$. Prove that $T$ is a compact operator.

6. (15) (D) Use the Fourier transform method to show that if $r = \{x, y, z\}$, then the equation

$$u_{xx} + u_{yy} + u_{zz} = -q(x, y, z),$$

where $q$ is a continuous, positive function on $\mathbb{R}^3$, has the formal solution

$$u(r) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{q(s)}{\|s-r\|} ds.$$