PhD Qualify Exam, PDE, Sep. 19, 2003

Show all works

1. Suppose that \( g \in C^1(R) \) and that \( M \equiv \sup_R |g(x)| < \infty \). Consider the initial value problem

\[
\begin{cases}
  u_t + u^2 u_x = 0 & \text{for } x \in R, \ t > 0 \\
  u(x,0) = g(x) & \text{for } x \in R.
\end{cases}
\]  

(a) Show that a \( C^1 \) solution \( u(x,t) \) satisfies \( |u(x,t)| \leq M \) for as long as the solution exists. [5%]

(b) Suppose that at some \( x_0 \in R, \ g(x_0)g'(x_0) < 0 \). Show that a \( C^1 \) solution breaks down in finite time. (Hint: study the behavior of \( u_x \) along characteristics.) [5%]

2. Let \( u \) be a solution of the wave equation in all of \( R^3 \times R \). Suppose that \( a > 0 \) and that \( u(x,0) = u_t(x,0) = 0 \) for \( |x| \geq a \).

(a) Show that \( u(x,t) = 0 \) in the double cone \( |x| \leq |t| - a \) for \( |t| \geq a \). [5%]

(b) Show that there is a constant \( C > 0 \) such that

\[
\int_{R^3} u^2(x,t) \, dx \leq C, \quad \text{for all } t > 0.
\]

(Hint: Show that there is a finite energy solution of \( w_{tt} - \Delta w = 0 \) such that \( w_t = u \).) [5%]

3. Let \( u \) be a nonnegative harmonic function in a ball \( B_R(0) \). Show that for \( |x| < R \), [10%]

\[
\frac{R^{n-2}(R - |x|)}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n-2}(R + |x|)}{(R - |x|)^{n-1}} u(0).
\]

4. Classify all the linear transformations \( T \) under which the Laplacian is invariant, [10%]

\[
\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.
\]

5. Suppose that \( \Omega \) is an open bounded subset of \( R^n \), with smooth boundary. Let \( u(x,t) \) be a smooth solution of the problem

\[
\begin{cases}
  u_t - \Delta u + c(x,t)u = 0, & \text{in } \Omega \times (0,\infty), \\
  u = 0, & \text{on } \partial \Omega \times [0,\infty), \\
  u(x,0) = g(x), & \text{in } \Omega.
\end{cases}
\]  

(2)
(a) Suppose that \( g(x) \geq 0 \) and \( |c(x,t)| \leq K \), for all \((x,t) \in \overline{\Omega} \times [0,\infty)\). Show that \( u(x,t) \geq 0 \) for all \((x,t) \in \overline{\Omega} \times [0,\infty)\). [10%]

(b) Can you deduce that \( |u(x,t)| \) remains bounded as \( t \to \infty \)? [10%]

6. (a) Use the method of characteristics to solve

\[
\begin{align*}
 u_y + (u_x)^4 &= 0, \\
 u(x,0) &= \frac{3}{4} x^\frac{3}{4}.
\end{align*}
\]

What is the domain of existence of the solution? [10%]

(b) Is it possible to apply the Local Existence Theorem to the problem

\[
\begin{align*}
 u_y + (u_x)^4 &= 0, \\
 u(x,x) &= x, \quad \text{for} \quad x > 0, \\
 u_x(x,x) &= \frac{1}{\sqrt[4]{4x}} \quad \text{for} \quad x > 0.
\end{align*}
\]

Justify your answer. [10%]

7. Let \( u, v \in C^1(\overline{\Omega}) \) be conjugate harmonic functions, i.e., \( u_x = v_y \) and \( u_y = -v_x \), in a simply connected domain \( \Omega \) with \( C^1 \) boundary in \( \mathbb{R}^2 \). Show that on the boundary curve \( \partial \Omega \),

\[
\frac{du}{dn} = \frac{dv}{ds}, \quad \frac{dv}{dn} = -\frac{du}{ds},
\]

where \( \frac{d}{dn} \) denotes differentiation in the direction of the outer normal and \( \frac{d}{ds} \) differentiation in the counter-clockwise tangential direction. Show that these relations can be used to reduce the Neumann problem for \( u \) to the Dirichlet problem for \( v \). [10%]

8. Let \( \Omega \subset \mathbb{R}^n \) be open. Show that if there exists a function \( u \in C^2(\overline{\Omega}) \) vanishing on \( \partial \Omega \) for which the quotient

\[
\frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}
\]

reaches its infimum \( \lambda \), then \( u \) is an eigenfunction for the eigenvalue \( \lambda \), so that \( \Delta u + \lambda u = 0 \) in \( \Omega \). [10%]