1. Let $X$ be a random variable with finite second moment, and $\mu$ be a median of $X$.
Prove $|EX - \mu| \leq \sqrt{2}\text{Var}X$ \hspace{1cm} (10%)

2. Let $X, Y$ be independent random variables with common $N(0, 1)$ distribution.
Prove $X + Y$ and $X - Y$ are independent. \hspace{1cm} (10%)

3. Let $X, Y$ be iid random variables which have finite variance and if $Z_1 = X + Y,$
$Z_2 = X - Y$ are independent, prove that $X, Y, Z_1, Z_2$ are normal distributed. \hspace{1cm} (10%)

4. Let $X, Y$ be independent random variables. Prove $X + Y$ is normal distributed if
and only $X, Y$ are both normal. \hspace{1cm} (10%)

5. Prove that if $X_1, X_2, \ldots, X$ is a martingale, then for every $\varepsilon > 0$,
\[
P\left(\sup_n |X_n| > \varepsilon \right) \leq \frac{1}{\varepsilon} \int_{\{|X| > \varepsilon\}} |X| dP \leq \frac{1}{\varepsilon} E|X|. \hspace{1cm} (10%)
\]

6. For sequence of r.v.'s $\{X_n\}$, if $\lim_{n \to \infty} ES_n = 0$ where $S_n = \sum_{i=1}^{n} X_i$. Prove
(i) $\frac{S_n - ES_n}{n} \to 0$ in probability.
(ii) $\frac{S_n - ES_n}{n}$ not necessarily converges almost surely. \hspace{1cm} (20%)

7. Suppose $\{X_n\}$ is a sequence of iid symmetric r.v.'s \hspace{1cm} (20%)
(a) Show that for a sequence $\{a_n\} \subset \mathbb{R}$, $\sum_{k=1}^{\infty} a_kX_k$ converge almost surely as
\[n \to \infty, \text{ then } \sum_{k=1}^{\infty} a_k^2 < \infty. \]
(b) Give an example to show that $\sum_{k=1}^{\infty} a_k^2 < \infty$ is not sufficient. (Hint: use
\[a_n = \frac{1}{n^\alpha} \text{ and } X_1 \text{ with } E\left[|X_1|^{\frac{1}{\alpha}}\right] = +\infty) \]

8. Suppose that $\{Z_n\}$ denotes the $n$ coin tossing results for a fair coin. Show that
there is a constant $c$, independent of $n$ and $t$ so that $P\left(\max_{k \leq n} \frac{Z_k}{\sqrt{n}} \geq t\right) \leq \frac{c}{t^2}$. \hspace{1cm} (10%)